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Destroying Tricommutativity of a 3-Cube Using the Symmetric Functor

Edward G. Estrada

DESTROYING TRICOMMUTATIVITY OF A 3-CUBE USING THE SYMMETRIC FUNCTOR

A Thesis

by

EDWARD GRANT ESTRADA

Submitted to Texas A&M International University
in partial fulfillment of the requirements
for the degree of

MASTERS OF SCIENCE

August 2018

Major Subject: Mathematics

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ABSTRACT

Destroying Tricommutativity Of A 3-Cube Using The Symmetric Functor (August 2018)

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Chair of Committee: Dr. David K. Milovich

In the following thesis, in pursuit of a Master's degree in Mathematics at Texas A &M International University (TAMIU), Laredo, Texas, we provide a modicum of background and introductory material to motivate the search for, and demonstrate examples of a satisfactory example of a tricommutative 3-cube whose tricommutativity can be "destroyed" through the employ of the symmetric square functor SP^2. The successful identification of a counter example of such a 3-cube is done using projection maps on inverse limit systems and through the consideration of commutative diagrams of symmetric powers of spaces of weight 2^{ω_2}. A previous counterexample of destroyed tricommutativity involving the Vietoris hyperspace operator, also known as the exponential functor, was previously demonstrated through the work of Mr. R. Montemayor, and this work extends the result in a topological context to the case of the symmetric functor. The existence of both such cases was proven by Dr. D. K. Milovich in other previous work. Much of the background draws from the previous literature of the advisor to the author, Dr. D. K. Milovich, the text and papers of L. Heindorf, L. Shapiro.

Acknowledgements

I would like to express great appreciation towards Dr. David K. Milovich for allowing me to pursue a research interest closely related to his current work. Dr. Milovich has been absolutely instrumental to my completion of this work and an amazing mentor at every turn, with a nearly infinite capacity for patience, availability, and possessing a breadth of knowledge to stimulate interest in any mathematical subject I presented to him. I would also like to acknowledge and thank Dr. Goonatilake for a providing an accommodating administrative presence throughout my pursuit of this Master's thesis. I thank Dr. Al-Tameemi for providing guidance as a member of my committee, as well as inside and outside classes. I thank Dr. Hinojosa for participating in the committee. Finally I thank the entire administration and faculty of Texas A&M International University for cultivating the academic environment in which I was able to pursue this degree.

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1 Introduction

For the pursuit of a Master's thesis in Mathematics, with the assistance of Professor David Milovich, we would like to investigate a result drawing heavily from some previous achievements of Scepin [8], Milovich [5], and Montemayor [6]. In Montemayor's thesis, available through the TAMIU Library collection of dissertations, it was shown previously that there exists an example of a Boolean algebra where application of the exponential functor destroyed innate tricommutativity. We seek to find an example in this respect, but using the symmetric functor instead. The work of Montemayor was completed in the language of Boolean algebras, using subalgebras and atoms. Instead we will use topological language, in an attempt to reduce the number of sets for which tricommutativity must be verified.

This thesis written reflecting the model of *Notices of the American Mathematical Society.*

2 Topological Background

Firstly we recall a few basic topologies which will be useful to our study and some previous results we seek to make use of.

Definition 2.1 (Product Topology)**.** *In a Cartesian product set* $\prod_{\beta \in J} X_\beta$ *with a topology for each* X_β*, Let* S_β *denote the collection* $S_\beta = \{\pi_\beta^{-1}(U_\beta) \mid U_\beta$ *is open and* $U_\beta \subseteq X_\beta\}$*, and let* $\mathcal{S}$ *denote the union of these collections,* $\mathcal{S} = \bigcup_{\beta \in J}$*. Then this subbasis generates the product topology. In this, the product topology, we call the Cartesian product of an indexed family of topological spaces* $\prod_{\alpha \in J} X_\alpha$ *a product space [7, §19].*

Definition 2.2 (Quotient Topology)**.** *Let* X *and* Y *be topological spaces; let* $p : X \to Y$ *be a surjective map.* p *is a quotient map if for* $U \subset Y$*,* U *is open iff* $p^{-1}(U)$ *is open in* X*.*

2.1 Cardinality and Countability

Two sets A and B can be said to have the same size or *cardinality* given that there is a bijection from A to B. Ordinals are transitive set x such that $z \in y \in x \Rightarrow z \in x$, and well-ordered by $\in$. Well ordering means a linear ordering such that every nonempty set has a minimum. The class of ordinals is itself well-ordered by $x \leq y \Leftrightarrow x \subset y$. By Zermelo's Theorem (every set can be well-ordered), the Axiom of Choice implies that every set is in bijection with an ordinal. Denote $|A|$ as the least ordinal in bijection with $|A|$. This is the size of A, sometimes called the cardinality. This is to say: *Cardinality* or *size* of $A = |A|$. Additionally, if there is an injective function from A into B, A has size at most equal to the size of B.

for an injective function, f with $f : A \to B$,

$$|A| \leq |B|$$

Furthermore,

$$|A| = |B| \text{ iff both } |A| \leq |B| \text{ and } |B| \leq |A|.$$

This is the Schroeder-Bernstein Theorem, which is true even without Choice if we define the cardinality ordering using only injections and bijections. Moving beyond simple size comparison, we would like to assign labels to the different sizes of sets. We can employ the notions of countability, uncountability and ordinals to this effect. For each cardinality there is a least ordinal with that cardinality. Such ordinals are called cardinals. We sometimes use the term weight to refer to the least of the sizes of bases of a given space.

Definition 2.3 (Countable). *For a nonempty set A, if there is a surjective mapping* $\mathbb{N} \mapsto A$ *for* $\mathbb{N}$ *the naturals, A is countable. We also define* $|\mathbb{N}| = \omega_0$, *(*ω_0 *sometimes written* ω*). A is finite iff* $|A| < \omega_0$*. A is countable iff* $|A| \leq \omega_0$.

If no such surjective mapping exists, A is *uncountable*. A good example to make use of when forming an intuition for uncountability is the canonical Cantor set, also known as the Cantor middle-thirds set.

Definition 2.4 (Cantor Set). *Let* A_0 *be the closed interval [0,1] in* $\mathbb{R}$*. Let* A_1 *be the set obtained from* A_0 *by deleting the set* $(\frac{1}{3}, \frac{2}{3})$*. Let* A_2 *be the set obtained from* A_1 *by deleting the set* $(\frac{1}{9}, \frac{2}{9})$ *and* $(\frac{7}{9}, \frac{8}{9})$*. For* A_n *define* $A_n = A_{n-1} - \bigcup_{k=0}^{\infty}(\frac{1+3k}{3^n}, \frac{2+3k}{3^n})$*. Then the intersection* $C = \bigcap_{n\in\mathbb{Z}_+} A_n$ *is the Cantor set.[7, §27]*

This set is compact and Hausdorff. Furthermore, the Cantor middle-thirds set is homeomorphic to the product space 2^ω However, as 2^{ω_1} is *uncountable*, the Cantor set is not homeomorphic to 2^{ω_1}. 2^ω and 2^{ω_1} are algebraically interesting as their clopen algebras are (up to isomorphism) the unique free Boolean algebras of size ω and ω_1 respectively.

Definition 2.5. *Let* α^+ *be the least cardinal greater than* α*. Then* κ *is a successor cardinal iff* $\kappa = \alpha^+$ *for some* α*.* κ *is a limit cardinal iff* $\kappa > \omega$ *and is not a successor cardinal.*

If A is uncountable, with cardinality $|A| = \omega_{0+1}, |A| = \omega_1$, has uncountable weight. ω_α is the α-th uncountable cardinal. Successor cardinals of ω_α are $\omega_{\alpha+1} = \omega_\alpha^+$. Define these as below.

Definition 2.6. *Let* $\aleph_\alpha = \omega_\alpha$ *be the* $\alpha - th$ *uncountable cardinal defined by transfinite recursion on* α *by:*

1. $\omega_0 = \omega$

2. $\omega_{\alpha+1} = (\omega_\alpha)^+$

3. *For* γ *a limit,* $\omega_\gamma = \sup\{\omega_\alpha : \alpha < \gamma\}$.

2.2 Retract

Definition 2.7 (Retract)**.** *X is a retract of Y, if there exists a mapping r such that* $r\colon Y \to X$ *r is continuous and there exists a mapping e such that* $e : X \to Y$ *is continuous and the following diagram commutes.*

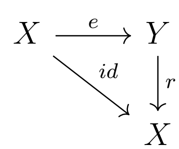

Figure 1: Commutative diagram of X as a retract of Y.

Definition 2.8 (Zero-dimensional)**.** *If a topological space has a basis consisting of only clopen sets, it is zero dimensional*

Definition 2.9 (Boolean Space)**.** *If a space is compact and Hausdorff and zero-dimensional, it is a Boolean space.*

Definition 2.10 (Free Boolean Algebra)**.** *Let X be an arbitrary set. A free Boolean algebra over X is a pair (e, F) such that F is a Boolean algebra and e is a map from X into F such that for every map f from X into a Boolean algebra A there is a unique homomorphism g: F* $\to$ *A satisfying* $g \circ e = f$*, that is, such that the following diagram commutes, [3, §9.1].*

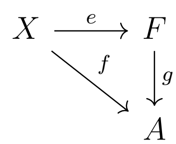

Figure 2: Commutative diagram of a Free Boolean Algebra.

A Boolean algebra is projective if and only if it is a retract of a free Boolean algebra. In this case, one can "dualize" the topological definition of retract by replacing *continuous* with *homomorphic* instead. For more on this see [9].
It is also useful to consider the category Fin,

- Fin: the category of finite sets with finite functions as the morphisms between them.

Considering finite sets and finite functions will be useful later. The specific subclass of Boolean spaces we limit our focus to are the retracts of powers of 2. These spaces are the Stone duals to *projective* Boolean algebras, and are also known as *absolute extensors of dimension 0* or $AE(0)$ or *Dugundji spaces.* The Stone representation theorem allows us to discuss characteristics of Boolean algebras in terms of their logically equivalent topological Stone duals, and vice versa, and is in some cases mechanically advantageous. In terms of categories, if we consider

- Bool: the category of Boolean algebras and Boolean homomorphisms
- Cpct_0: the category of Boolean spaces and continuous functions between them

Stone representation allows us to identify dualities between diagrams in Bool and diagrams in Cpct_0:

$$\text{Bool} \xleftrightarrow{\text{Stone Representation}} \text{Cpct}_0$$

For a given Boolean algebra B, denote 2^B the set of homomorphisms from B to 2. Then 2^B is a Stone space. For any given Stone space X, denote the family of all clopen sets as the algebra dual to X. For more this duality, see [9].

Definition 2.11 (Absolute Extensor of Dimension Zero)**.** *A Boolean space X is an* absolute extensor of dimension zero (AE(0)) *if it is an injective object in $Cpct_0$. In other words, if for every Boolean space Y, there is Boolean space Z such that $Z \subset Y$ with Z closed, and continuous $f : Z \to X$, there is a continuous $g : Y \to X$ extending f.*

3 Commutativity

Given a commutative square of functions $W \underset{f}{\leftarrow} X \underset{h}{\leftarrow} Z \underset{k}{\rightarrow} Y \underset{g}{\rightarrow} W$ as in the commutative diagram provided in Figure 3.

$$\begin{array}{ccc} X & \underset{h}{\longleftarrow} & Z \\ \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle k} \\ W & \underset{g}{\longleftarrow} & Y \end{array}$$

Figure 3: A simple commutative square.

The commutative square is said to *bicommute*, or be *bicommutative* if, for every pair (x, y) such that $f(x) = g(y)$, there exists z such that $(h, k)(z) = (x, y)$.

Definition 3.1 (Relatively Complete). *Given a Boolean algebra A, suppose there are two finite sets $U(b) \subseteq \{c \in A : b \leq c\}$ and $L(b) \subseteq \{c \in A : c \leq b\}$ such that, if $a \leq b$, then $U(a) \cap L(b) \neq \emptyset$. Then A is said to be relatively complete.*

It is a classical result of Scepin [8] that all Boolean algebras up to size $\aleph_1$ are relatively complete. We say also that if a Boolean Algebra is relatively complete, then it is *rc-filtered*.

Definition 3.2. *A poset (or partially ordered set) is a set with a partial ordering relation applied to it.*

The defining properties of a poset, $(I, \leq)$:

$$\text{for all } x, y, z \in I,$$

$$x \leq x,$$

$$x \leq y \leq z \Rightarrow x \leq z,$$

$$x \leq y \leq x \Rightarrow x = y,$$

A poset I is *directed* if for every $x, y \in I, \exists z \in I$ such that $x, y \leq z$.

Definition 3.3 (Meet-semilattice)**.** *A poset is a meet semilattice if every two-element set* $\{i, j\}$ *has a greatest lower bound* $i \wedge j$*.*

Definition 3.4 (Inverse limit system)**.** *An inverse limit system indexed by a directed meet-semilattice* $(I, \leq)$ *consists of continuous surjections*

$$f_i^j : X_j \to X_i \textit{ for all } i \leq j$$

$$\textit{such that } f_i^j \circ f_j^k = f_i^k, \textit{ for all } i \leq j \leq k$$

with each f_i^i *the identity map on* X_i, $f_i^i : X_i \to X_i$*.*

Definition 3.5 (Inverse Limit)**.** *For a family of spaces,* $\{X_n\}$ *an* inverse limit *consists of*

$$X_\infty = \left\{ p \in \Pi_{i \in I} X_i \middle| \forall (i \leq j) \quad p(i) = f_i^j(p(j)) \right\}$$

and the projection maps $f_i^\infty : X_\infty \to X_i$ *where* $f_i^\infty(p) = p(i)$ *such that* $f_i^\infty = f_i^j \circ f_j^\infty$*.*

The projection maps are called *projections* because they are restrictions of coordinate projections. The inverse limit X_∞ has a basis consisting of sets of the form $(f_i^\infty)^{-1}(U)$ for U open, $U \subset X_i$.

4 Motivation

The symmetric power functor has surprising properties when it comes to how it handles $AE(0)$ spaces.

Definition 4.1 (Symmetric Power functor)**.** *Given a topological space X the n-th symmetric power $SP^n(X)$ is the quotient of the product topology (X^n) with respect to the right action of $S_n, xg = x \circ g$ for all $x : n \to X$ and all bijective $g : n \to n$. In the square case, $(a, b)/S_2 = (b, a)/S_2$ for all $a, b \in X$.*

In which cases does this functor map a bicommutative subdiagram to another bicommutative subdiagram? Furthermore, when does the symmetric power functor generate tricommutative or non-tricommutative subdiagrams? Are there any anomalies such as non-tricommutative subdiagrams within maps of the symmetric power functor applied to previously tricommutative diagrams?

Theorem 4.1. *For a topological space X, given $0 < m < n < \omega$, and $\kappa \leq \omega_1$,*
$2^\kappa \cong SP^n(2^\kappa)$ and $2^\kappa \cong SP^m(2^\kappa) \cong SP^n(2^\kappa)$ so $Sp^n(2^\kappa)$ is a retract of 2^κ.

Remember that ω_1 is the first uncountable ordinal.

Theorem 4.2. *For $1 \leq m < n < \omega$, $\kappa \geq \omega_2$, $SP^n(2^\kappa)$ is not a retract of $SP^m(2^\kappa)$.*

Now we can observe a place where properties conserved by the Symmetric functor do not scale, or at least start to differ with increasing cardinality. In the case $m = 1$, $\kappa = \omega_2$,

$$SP^n(2^{\omega_2}) \text{ is not a retract of } 2^{\omega_2}.$$

This result is due in large part to Scepin [8], [2, Corrollary 3.4.8]. Scepin additionally proved that this breaks down past size ω_2, a motivating result. Why should the symmetric power of 2^{ω_2} fail to be a retract of 2^{ω_2} when the ω_1 case does not fail? This was answered in terms of inverse limits of continuous surjections between Cantor sets by Scepin, while

Milovich answers this in terms of inverse limits of finite surjections, yet non-constructively. Milovich's answer leaves room for the construction of an example of a tricommutative 3-cube with a non-tricommutative subdiagram using Boolean spaces, Fin, and S^2.

To accomplish this, we begin by considering bicommutativity amongst Boolean spaces (objects of Bool). Given a family of Boolean spaces X_i consider an inverse limit system, $f_i^j : X_j \to X_i$ for all $i \leq j$ in some directed meet-semilattice, and every quadruple (h, i, j, k) with the ordering $h \leq i \leq k \geq j \geq h$. This system can be seen as the below commutative square in Figure 4,

$$\begin{array}{ccc} X_i & \xleftarrow{f_i^k} & X_k \\ {\scriptstyle f_h^i}\downarrow & & \downarrow{\scriptstyle f_j^k} \\ X_h & \xleftarrow[f_h^j]{} & X_j \end{array}$$

Figure 4: A bicommutative diagram.

One may ask, do all such squares bicommute for all such h, i, j, k with $h = i \wedge j$? If they do, we say that the inverse limit of this system, X_∞, is a bicommutative limit of the $X_i's$. In their monograph, Heindorf and Shapiro [2, Lemma 4.3.5] prove an interesting lemma in algebraic terms concerning the topological equivalence of bicommutative squares and commuting subalgebras.

Lemma 4.3. *Assume that A and B are commuting subalgebras of some Boolean algebra C. Then the symmetric powers $SP^2(A)$and $SP^2(B)$ are commuting subalgebras of $SP^2(C)$.*

With inclusions as morphisms we are able to swap commuting subalgebras to bicommutative squares by Stone duality. This then lets us state that the symmetric functor preserves bicommutativity. This is a critical fact in our motivation, as we shall see later.

Theorem 4.4. *Every 2^κ is a $(<\omega)-$commutative inverse limit of finite surjections [5].*

In regards to our bicommutative diagram above, we define an *$n -$ cube subdiagram.*

Definition 4.2 (n-cube subdiagram)**.** *Let* $f_i^j : X_j \to X_i$ *for all* $i \leq j$ *in* $(I, \leq)$*, a directed meet-semilattice. An n-cube subdiagram is a commutative n-cube of the form* $f_{\varphi(\sigma)}^{\varphi(\tau)} : X_{\varphi(\tau)} \to X_{\varphi(\sigma)}$*, for* $\tau \subset \sigma \subset n$ *for some* $\wedge$*-preserving map* $\varphi : (\mathcal{P}(n), \supset) \to (I, \leq)$*.*

An extension from bicommutativity to *tricommutativity* can also be defined.

Definition 4.3 (Tricommutative)**.** *For a commutative cube* $f_\sigma^\tau : X_\tau \to X_\sigma$*, with* $\tau \subset \sigma \subset 3$*, the cube* f_σ^τ *is* tricommutative *if for every* $(x_{\{0\}}, x_{\{1\}}, x_{\{2\}})$ *such that* $f_{\{i,j\}}^{\{i\}}(x_{\{i\}}) = f_{\{i,j\}}^{\{j\}}(x_{\{j\}})$ *for all* $i < j < 3$ *there exists* $x_\varnothing$ *such that* $f_{\{i\}}^{\varnothing}(x_{\{i\}}) = x_\varnothing$ *for all* $i < 3$*.* $\varnothing$ *denotes the empty set as an index placeholder.*

Going further, from bicommutative and tricommutative to n-commutative, say that an inverse limit system $n-$*commutes* if all its n-cube subdiagrams n-commute. If all the n-cube subdiagrams n-commute for all $n < \omega$ the inverse limit system $(< \omega)$ commutes. From [5, §3] again we have that,

Theorem 4.5. *For* $n \geq 2$*,* $SP^n(2^{\aleph_2})$ *is not a tricommutative limit of finite surjections.*

With these two results, it must be true that there exists at least one example of SP^n in which tricommutativity is destroyed. The identification of at least one sufficient example in this regard will be the main focus of our study. It was shown by Milovich and Montemayor [6] that there exists a 3-cube in the category Fin whose 3-commutativity is destroyed by another functor called the Vietoris hyperspace operator, *Exp*. We will not define *Exp*, for more on this see [6] and [5]. Monetmayor managed to find a sufficient example of a a 3-commutative 3-cube whose 3-commutativity was destroyed cube using: After applying *Exp*, the exponential functor, he found 3-commutativity destroyed as in Figure 6. This work was done primarily in the language of Boolean algebras. Each vertex in these diagrams in such a case represents an atom of a Boolean algebra. Milovich has translated this example into a topological context. In a more direct explanation, the three sets {12,3,4}, {1,4}, {1,34} represent compatible points in three spaces produced by the *Exp* acting on the earlier cube. A pair of such sets in this context is compatible if they agree on the maps f & g, h & i, or

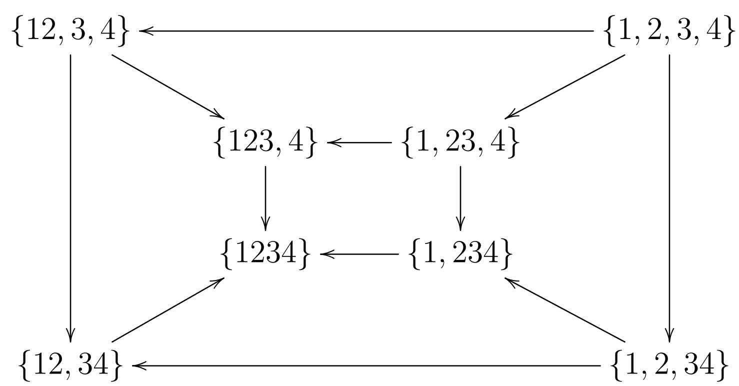

Figure 5: Montemayor's tricommutative diagram.

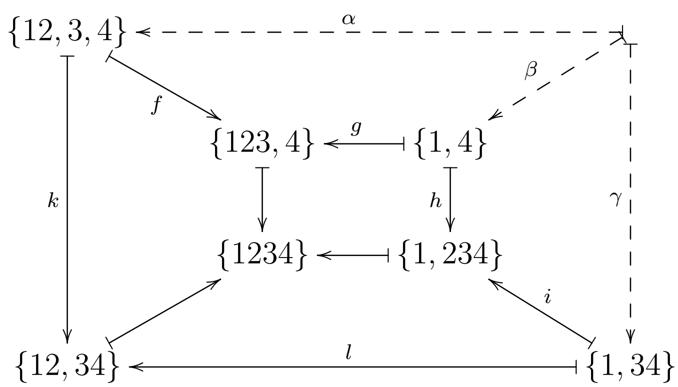

Figure 6: A non-tricommutative subdiagram following application of *Exp*.

k & l simultaneously. The 3-commutativity is broken as there does not exist a set which reaches {12,3,4}, {1,4}, {1,34} simultaneously using the maps α, β, and γ. See [6] for an entire discussion on this in a Boolean algebra context. We seek a similar counterexample but in topological terms and using the symmetric functor as opposed to the *Exp*.

Proposition 4.1 (Main Proposition)**.** *There exists a 3-commutative 3-cube in the category Fin whose objects are small finite sets and whose 3-commutativity is destroyed by* SP^2.

We propose to find at least one example of such a 3-cube in the category of finite objects which is already tricommutative, then apply the symmetric square functor to it, and find a counterexample to tricommutativity in the result of the transformation by the functor. As the number of subdiagrams scales almost quadratically up to symmetry with the starting diagram, this may require an exhaustive search.

5 Results on Montemayor's 3-Cube

Here we outline the method for performing tests on specific examples of tricommutative 3-cubes. Using commutative diagrams we check what we term *compatibility* on specific pairs of points following mappings after the application of S^2.

We will first attempt to break tricommutativity by applying the symmetric functor to the 3-cube that was presented by Montemayor as a counterexample. The 3-cube used previously is as in Figure 7:

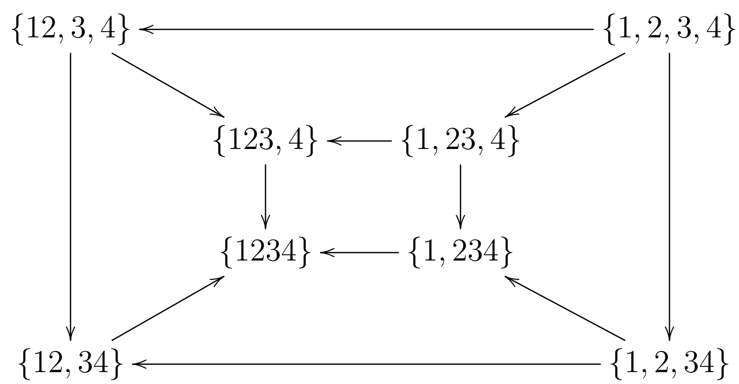

Figure 7: Montemayor's 3-Cube.

We apply S^2 to this. This maps as:

$$\{1,2,3,4\} \longrightarrow \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}.$$

An empirical example of the symmetric pairing effects of S^2. Next, We identify each mapping in the commutative diagram as in Figure 8.

We call two points a, b in the "second tier" of the diagram if they exist after the application of S^2 and after the mapping of the colored maps, red, green, or blue. We call a set of points *compatible* if they agree on either ι, κ,or θ. That is, a and b are compatible if $\theta(a) = \theta(b)$. We may extend this notion compatibility to triples using the below definition, and this will turn out to be the main focus of the study.

Definition 5.1 (Compatible Triple)**.** *For three points a, b, c in the "second hierarchy" of the diagram (the image of the color maps), $a \in im(red)$, $b \in im(blue)$, $c \in im(green)$, a, b, and c are a compatible triple if $\theta(a) = \theta(b)$, $\iota(b) = \iota(c)$, and $\kappa(a) = \kappa(c)$.*

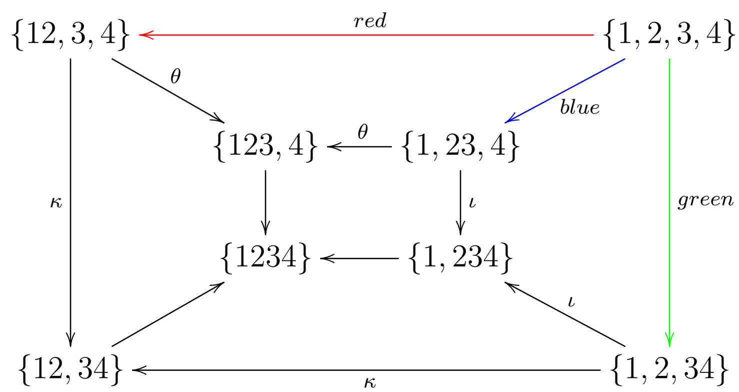

Figure 8: Montemayor's 3-Cube with mappings colored and labeled.

6 Compatibility Results

Any three points found to be compatible, but not simultaneously in any image of the origin set: $S^2\{1,2,3,4\}$, under the "color maps", destroys tricommutativity. In other words, if there exists a compatible triple (a,b,c) but there does not exist $z \in S^2\{1,2,3,4\}$ such that $red(z) = a$, $blue(z) = b$, $green(z) = c$, then (a,b,c) is a sufficient counterexample destroying the tricommutativity of our 3-cube.

6.1 Montemayor's 3-Cube

Using these concepts, we proceed to check all sets of triples for the existence of a satisfactory counterexample within the cube given by Montemayor [6], which was shown to have a counterexample in the *Exp* functor case. The initial mapping from the symmetric square functor on $\{1,2,3,4\}$ can be tabulated as follows, with the numbers in brackets $\{\}$ denoting the associated color map applied to each of the points from the functor in the left column:

	$\{12,3,4\}$	$\{1,23,4\}$	$\{1,2,34\}$
1,1	12,12	1,1	1,1
1,2	12,12	1,23	1,2
1,3	12,3	1,23	1,34
1,4	12,4	1,4	1,34
2,2	12,12	23,23	2,2
2,3	12,3	23,23	2,34
2,4	12,4	23,4	2,34
3,3	3,3	23,23	34,34
3,4	3,4	23,4	34,34
4,4	4,4	4,4	34,34

Table 1: Diagonal Products Reference for Montemayor's 3-Cube.

We use Table 1 as our primary reference for checking *off-diagonal* triples, that is, any set of three points not in the same row simultaneously, then check compatibility of the triple after the mappings of θ, ι, or κ. This list is fairly extensive. Each set of triples has 3 associated checks required, but if the first check fails, we can discard the triple and move on to the next, shortening the process slightly. We chose to break this up into an iterative order, as will be shown, which helps to expedite how many triples can be discarded quickly. In these sets of tables, the bracketed sets denote the θ, ι, κ maps. The cells in the table with the double pairs of three digit points, the pair on the left corresponds to the point in the left column under the associated map, while the right pair corresponds to the point in the column towards the left under the associated map. For example in the first row, $1, 1 \mapsto_\iota (1, 1)$ (1 mapped to 1), while $1, 2 \mapsto_\iota (1, 234)$ (1 mapped to 1, 2 mapped to 234). In the sixth column, we have the mappings of the point from the first column and the point from the fifth column, in opposite order. We are looking explicitly for compatibility across all rows, while making sure all three points we map are not in any row of our reference table simultaneously. If any pair of double three digit points disagree, the triple fails as a sufficient counterexample. The triples we check reside in the first, third, and fifth rows. We procedurally fix and vary the points across these columns.

	$\theta : \{123, 4\}$		$\iota : \{1, 234\}$		$\kappa : \{12, 34\}$
12,12	(123,123),(123,123)	1,1	(1,1),(1,234)	1,2	(12,12),(12,12)
12,12	(123,123),(123,123)	1,1	(1,1),(1,234)	1,34	(12,12),(12,34)
12,12	(123,123),(123,123)	1,1	(1,1),(234,234)	2,2	(12,12),(12,12)
12,12	(123,123),(123,123)	1,1	(1,1),(234,234)	2,34	(12,12),(12,34)
12,12	(123,123),(123,123)	1,1	(1,1),(234,234)	34,34	(12,12),(34,34)
12,12	(123,123),(123,123)	1,23	(1,234),(1,1)	1,1	(12,12),(12,12)
12,12	(123,123),(123,123)	1,23	(1,234),(1,1)	1,34	(12,12),(12,34)
12,12	(123,123),(123,123)	1,23	(1,234),(234,234)	2,2	(12,12),(12,12)
12,12	(123,123),(123,123)	1,23	(1,234),(234,234)	2,34	(12,12),(12,34)
12,12	(123,123),(123,123)	1,23	(1,234),(234,234)	34,34	(12,12),(34,34)
12,12	(123,123),(123,4)	1,4	(1,234),(1,1)	1,1	(12,12),(12,12)

Table 2: Compatibility Checks for (12,12), (1,1), (1,2) through (12,12), (1,4), (34,34).

We cannot observe three point compatibility for any $\theta((12, 12), (1, 4), n)$ for any $n \in S^2\{1, 2, 3, 4\}$ so we can skip the checks of the maps ι and κ. We finish Table 2 with Table 3 starting at $(12, 12)$ and $(23, 23)$.

	$\theta : \{123, 4\}$		$\iota : \{1, 234\}$		$\kappa : \{12, 34\}$
12,12	(123,123),(123,123)	23,23	(234,234),(1,1)	1,1	(12,12),(12,12)
12,12	(123,123),(123,123)	23,23	(234,234),(1,234)	1,2	(12,12),(12,12)
12,12	(123,123),(123,123)	23,23	(234,234),(1,234)	1,34	(12,12),(12,34)
12,12	(123,123),(123,123)	23,23	(234,234),(2,34)	2,34	(12,12),(12,34)
12,12	(123,123),(123,123)	23,23	(234,234),(2,34)	34,34	(12,12),(34,34)
12,12	(123,123),(4,4)	23,4			
12,12	(123,4),(4,4)	4,4			

Table 3: Compatibility Checks for (12,12), (23,23), (1,1) through (12,12), (23,23), (34,34).

Again, we fail to observe compatibility in the first column, between (12,12), (23,4) and (12,12), (4,4), so we can proceed directly without checking the rest of the triples involving those two pairs. We continue at (12,3), (1,1), (1,1). For the rest of these checks, we suppress discussion of when a check fails in either θ, ι, or κ and proceed directly, leaving space in the table where further checks are suppressed once an incompatibility is found. This also helps to highlight at which pairing a check fails. We continue, finishing the remainder of the compatibility checks on Montemayor's 3-Cube. Observing these checks beginning with Tables 4 and 5, we can see that there are no sufficient counterexamples to be found amongst the non-diagonal products. We may now observe with confidence that Montemayor's 3-Cube, which served to produce a sufficient counterexample of destroyed tricommutativity in a subdiagram using the Vietoris hyperspace functor, Exp, fails to produce a tricommutativity destroying subdiagram under the symmetric power functor, S^2. For this reason, we seek out other 3-cubes. Examples that fit our criteria need to possess innate bicommutativity as well as tricommutativity before the application of our functor of choice S^2. Furthermore, it would be more intuitive to search for examples of 3-cubes where symmetry amongst the potential triple point combinations we search through present with some form of natural symmetry, to better aid the manual tabulation of possible commutativity checks. Beyond an exhaustive amount of manual searches, should the lack of existence of a counterexample continue, the implementation of software may be required for more rigorous compatibility verification due to the quadratic scaling of subdiagrams generated.

12,3	(123,123),(123,123)	1,1	(1,1),(1,1)	1,1	(12,34),(12,12)
12,3	(123,123),(123,123)	1,1	(1,1),(1,234)	1,2	
12,3	(123,123),(123,123)	1,1	(1,1),(1,234)	1,34	
12,3	(123,123),(123,123)	1,1	(1,1),(234,234)	2,2	
12,3	(123,123),(123,123)	1,1	(1,1),(234,234)	2,34	
12,3	(123,123),(123,123)	1,1	(1,1),(234,234)	34,34	
12,3	(123,123),(123,123)	1,23	(1,234),(1,1)	1,1	
12,3	(123,123),(123,123)	1,23	(1,234),(1,234)	1,2	(12,34),(12,12)
12,3	(123,123),(123,123)	1,23	(1,234),(234,234)	2,2	
12,3	(123,123),(123,123)	1,23	(1,234),(234,234)	2,34	
12,3	(123,123),(123,123)	1,23	(1,234),(234,234)	34,34	
12,3	(123,123),(123,4)	1,4		1,1	
12,3	(123,123),(123,123)	23,23	(234,234),(1,1)	1,1	
12,3	(123,123),(123,123)	23,23	(234,234),(1,234)	1,2	
12,3	(123,123),(123,123)	23,23	(234,234),(1,234)	1,34	
12,3	(123,123),(123,123)	23,23	(234,234),(234,234)	2,2	(12,34),(12,12)
12,3	(123,123),(123,123)	23,23	(234,234),(234,234)	34,34	(12,34),(34,34)
12,3	(123,123),(123,4)	23,4			
12,3	(123,123),(4,4)	4,4			

Table 4: Compatibility checks for Montemayor's 3-Cube, $\{(12,3),(1,1),(1,1)\}$ through $\{(12,3),(4,4),(n,m)\}$ for $n,m \in S^2\{1,2,3,4\}$.

12,4	(123,4),(123,123)	1,1			
12,4	(123,4),(123,123)	1,23			
12,4	(123,4),(123,4)	1,4	(1,234),(1,1)	1,1	
12,4	(123,4),(123,4)	1,4	(1,234),(1,234)	1,2	(12,34),(12,12)
12,4	(123,4),(123,4)	1,4	(1,234),(234,234)	2,2	
12,4	(123,4),(123,4)	1,4	(1,234),(234,234)	2,34	
12,4	(123,4),(123,4)	1,4	(1,234),(234,234)	34,34	
12,4	(123,4),(123,123)	23,23			
12,4	(123,4),(123,4)	23,4	(234,1),(1,1)	1,1	
12,4	(123,4),(123,4)	23,4	(234,234),(1,234)	1,2	
12,4	(123,4),(123,4)	23,4	(234,234),(1,234)	1,34	
12,4	(123,4),(123,4)	23,4	(234,234),(234,234)	2,2	(12,34),(12,12)
12,4	(123,4),(123,4)	23,4	(234,234),(234,234)	34,34	(12,34),(34,34)
12,4	(123,4),(4,4)	4,4			

Table 5: Compatibility checks for Montemayor's 3-Cube, $\{(12,4),(1,1),(a,b)\}$ through $\{(12,4),(4,4),(n,m)\}$. for $a,b,n,m \in S^2\{1,2,3,4\}$.

So far, we observe no tricommutativity destroying subdiagrams amongst these point combinations. The checks are broken up into combinations iterated around a fixed point in the left-most column, with each table representing iterations on each fixed point. We finish searching for non-tricommutative subdiagrams in Montemayor's 3-Cube with Tables 6, 7, and 8.

3,3	(123,123),(123,123)	1,1	(1,1),(1,1)	1,1	(34,34),(12,12)
3,3	(123,123),(123,123)	1,1	(1,1),(1,1)	1,2	(34,34),(12,12)
3,3	(123,123),(123,123)	1,1	(1,1),(234,234)	1,34	
3,3	(123,123),(123,123)	1,1	(1,1),(234,234)	2,2	
3,3	(123,123),(123,123)	1,1	(1,1),(234,234)	2,34	
3,3	(123,123),(123,4)	1,4			
3,3	(123,123),(123,123)	23,23	(234,234),(1,1)	1,1	
3,3	(123,123),(123,123)	23,23	(234,234),(1,234)	1,2	
3,3	(123,123),(123,123)	23,23	(234,234),(1,234)	1,34	
3,3	(123,123),(123,123)	23,23	(234,234),(234,234)	2,2	(34,34),(12,12)
3,3	(123,123),(123,123)	23,23	(234,234),(234,234)	2,34	(34,34),(12,34)
3,3	(123,123),(123,4)	23,4			
3,3	(123,123),(4,4)	4,4			

Table 6: Compatibility checks for Montemayor's 3-Cube, $\{(3,3),(1,1),(1,1)\}$ through $\{(3,3),(4,4),(n,m)\}$. for $n,m \in S^2\{1,2,3,4\}$.

3,4	(123,4),(123,123)	1,1			
3,4	(123,4),(123,123)	1,23			
3,4	(123,4),(123,4)	1,4	(1,234),(1,1)	1,1	
3,4	(123,4),(123,4)	1,4	(1,234),(1,234)	1,2	(34,34),(12,34)
3,4	(123,4),(123,4)	1,4	(1,234),(1,234)	1,34	(23,23),(12,34)
3,4	(123,4),(123,4)	1,4	(1,234),(234,234)	2,2	
3,4	(123,4),(123,4)	1,4	(1,234),(234,234)	2,34	
3,4	(123,4),(123,4)	1,4	(1,234),(234,234)	34,34	
3,4	(123,4),(123,123)	23,23			
3,4	(123,4),(123,4)	23,4	(234,234),(1,1)	1,1	
3,4	(123,4),(123,4)	23,4	(234,234),(1,234)	1,2	
3,4	(123,4),(123,4)	23,4	(234,234),(1,234)	1,34	
3,4	(123,4),(123,4)	23,4	(234,234),(234,234)	2,2	(34,34),(12,12)
3,4	(123,4),(123,4)	23,4	(234,234),(234,234)	2,34	(23,23),(12,34)
3,4	(123,4),(4,4)	4,4			

Table 7: Compatibility checks for Montemayor's 3-Cube, $\{(3,4),(1,1),(a,b)\}$ through $\{(3,3),(4,4),(n,m)\}$. for $a,b,n,m \in S^2\{1,2,3,4\}$.

4,4	(4,4),(123,123)	1,1			
4,4	(4,4),(123,123)	1,23			
4,4	(4,4),(123,4)	1,4			
4,4	(4,4),(123,123)	23,23			
4,4	(4,4),(123,4)	23,4			
4,4	(4,4),(4,4)	4,4	(234,234),(1,1)	1,1	
4,4	(4,4),(4,4)	4,4	(234,234),(1,234)	1,2	
4,4	(4,4),(4,4)	4,4	(234,234),(1,234)	1,34	
4,4	(4,4),(4,4)	4,4	(234,234),(234,234)	2,2	(34,34),(12,12)
4,4	(4,4),(4,4)	4,4	(234,234),(234,234)	2,34	(234,234),(12,34)

Table 8: Compatibility checks for Montemayor's 3-Cube, $\{(4,4),(1,1),(a,b)\}$ through $\{(4,4),(4,4),(2,34)\}$. for $a, b \in S^2\{1,2,3,4\}$.

6.2 The Binary 8-Cube

We now attempt to find a counterexample where tricommutativity is destroyed in a separate 3-cube. This 3-cube consists of the numbers 0 through 7, with maps between permutations of alternating pairs of numbers. We refer to this cube as the Binary 8-cube as the maps may be easily seen as permutations of the digits translated to binary paired in an ascending, almost lexicographical order. The set $\{0,1,2,3,4,5,6,7\}$ may be represented in binary as $\{000,001,010,011,100,101,110,111\}$. Then we establish the first mappings (following S^2) θ : $\{(000,001),(010,011),(100,101),(110,111)\}$, $\iota : \{(000,010),(001,011),(100,110),(101,111)\}$, $\kappa : \{(000,100),(001,101),(010,110),(011,111)\}$. We can also see these maps as $\theta : \{01,23,45,67\}$, $\iota : \{02,13,46,57\}$, $\kappa : \{04,15,26,37\}$. Defining the rest of the cube, we have the following commutative diagram in Figure 9. This cube is tricommutative as per Milovich [5].

Our reference tables of diagonal products for compatibility checks then become Table 9 and Table 10. We break this reference up into two tables for readability. These subdiagrams form the basis for construction of counterexamples using off-diagonal triple point combinations.

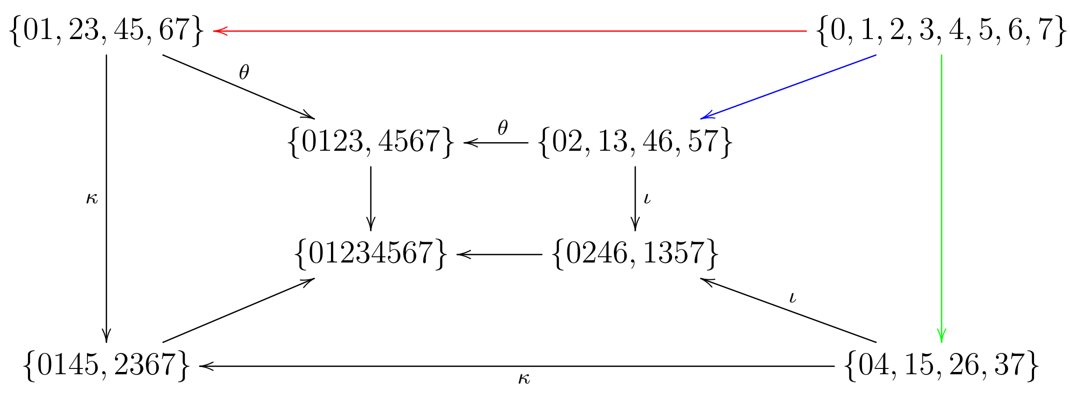

Figure 9: The Binary 8-Cube used to find a non-tricommutative subdiagram.

	$\theta : \{01, 23, 45, 67\}$	$\iota : \{02, 13, 46, 57\}$	$\kappa : \{04, 15, 26, 37\}$
0,0	01,01	02,02	04,04
0,1	01,01	02,13	04,15
0,2	01,23	02,02	04,26
0,3	01,23	02,13	04,37
0,4	01,45	02,46	04,04
0,5	01,45	02,57	04,15
0,6	01,67	02,46	04,26
0,7	01,67	02,57	04,37
1,1	01,01	13,13	15,15
1,2	01,23	13,02	15,26
1,3	01,23	13,13	15,37
1,4	01,45	13,46	15,04
1,5	01,45	13,57	15,15
1,6	01,67	13,46	15,26
1,7	01,67	13,57	15,37
2,2	23,23	02,02	26,26
2,3	23,23	02,13	26,37
2,4	23,45	02,46	26,04
2,5	23,45	02,57	26,15
2,6	23,67	02,46	26,04
2,7	23,67	02,57	26,37

Table 9: Diagonal Products Reference of the Binary 8-Cube for (0,0) through (2,7)

	$\theta:\{01,23,45,67\}$	$\iota:\{02,13,46,57\}$	$\kappa:\{04,15,26,37\}$
3,3	23,23	13,13	37,37
3,4	23,45	13,46	37,04
3,5	23,45	13,57	37,15
3,6	23,67	13,46	37,26
3,7	23,67	13,57	37,37
4,4	45,45	46,46	04,04
4,5	45,45	46,57	04,15
4,6	45,67	46,46	04,26
4,7	45,67	46,57	04,37
5,5	45,45	57,57	15,15
5,6	45,67	57,46	15,26
5,7	45,67	57,57	15,37
6,6	67,67	46,46	26,26
6,7	67,67	46,57	26,37
7,7	67,67	57,57	37,37

Table 10: Diagonal Products Reference of the Binary 8-Cube for (3,3) through (7,7).

These 2 tables (Table 9 and Table 10 combined) are significantly larger than our previous 10 row table, demonstrating how quickly the difficulty to perform checks for counterexamples scales with the number of elements in the Boolean space under consideration.

6.3 A Sufficient Counterexample

Consider the compatible triple: $\{(01,67),(02,57),(26,15)\}$ in our binary 3-cube. With the maps prescribed above, the points demonstrate compatibility across θ,ι, and κ. Performing our standard check for compatibility we observe a convenient finding:

01,67	(0123,4567),(0123,4567)	02,57	(0246,1357),(0246,1357)	26,15	(0145,2367),(0145,2367)

Table 11: Counterexample Compatibility Verification.

Using Table 11 we observe compatibility across three chosen points in a pairwise manner. We inspect our reference via Tables 9, 10 and note that the three points $(01, 67), (02, 57), (26, 15)$ do not occur in any row simultaneously, and therefore are not in any diagonal product of the elements of $S^2(\{0, 1, 2, 3, 4, 5, 6, 7\})$, and hence is a sufficient example destroying the tricommutativity of Binary 8-Cube.

7 Summary and Future Prospects

As shown in the checks above, Montemayor's 3-Cube failed to provide a sub-diagram breaking tricommutativity under the action of the symmetric functor. Inspection of the cube we moved to next, our tricommutative Binary 8-Cube under the action of the symmetric functor yielded the non-tricommutative subdiagram using the numbers (01,67), (02,57), (26,15). The subdiagram presents as below in Figure 10.

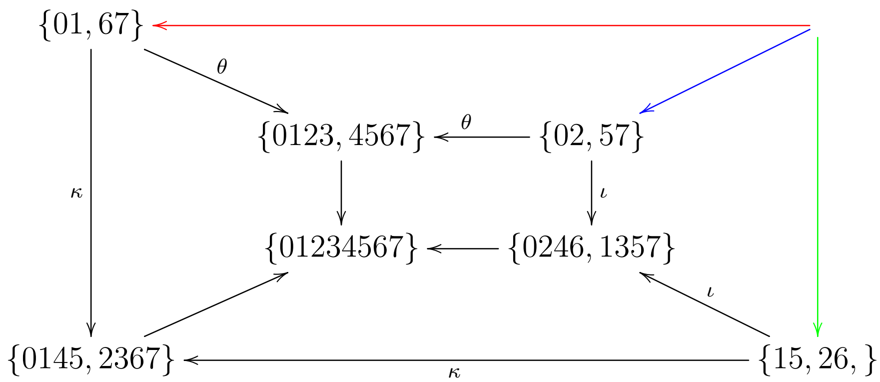

Figure 10: Our result, a non-tricommutative subdiagram in the Binary 8-Cube.

We may now say in confidence that tricommutativity may be destroyed using $SP^2(2^{\aleph_2})$, confirming the hypothesis of Milovich [5]. As Montemayor [6] already proved the existence of a similarly destructive subdiagram in the case of the Vietoris hyperspace operator Exp, both conjectures from Milovich [5] are shown to be accurate.

7.1 Beyond Tricommutativity

Looking towards potentially constructive work that could be done on these results, one may naturally wonder as to what other functors are able to destroy the tricommutativity in 3-cube tricommutative diagrams. Answering such a question would certainly make for interesting further research. Exploring n-commutativity in n-cubes beyond $n = 3$ is another way to extend these notions. Above the $n = 3$ level of commutativity, one might ask, do all n-commutative n-cubes possess a commutativity destroying subdiagram for a given functor, like our $n = 3$ case did for S^2 and Exp? One might also ask if it is only these functors which destroy tricommutativity, or n-commutativity in general. Extending considerations to commutative diagrams of arbitrary size may result in some lower limit on n, above which non-commutative subdiagrams might always exist. Observing how quickly we were able to deduce the presence of a non-commutative subdiagram in the Binary 8-Cube case, such a conjecture is plausible. Alternatively, had we been unfortunate and not found any example of tricommutativity destroyed in the first few hundred checks on the Binary 8-Cube, we would have resorted to automation of the checking process. Development of computer code to check arbitrary 3-cubes and potentially, n-cubes would make for prospective future research, lending confidence when the required checks scales into the thousands, millions, and so on, as n grows.

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