1-25-2018

Breaking the Weak Commutative Property of Finite Boolean Subalgebra Triples

Rene Montemayor

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A Thesis

by

RENE MONTEMAYOR

Submitted to Texas A&M International University
in partial fulfillment of the requirements
for the degree of

MASTER OF SCIENCE

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Approved as to style and content by:

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DEDICATION

I would like to dedicate this thesis to my family for their support and understanding during so many long hours spent away from them in trying to finalize this work. I wish to thank in a very special way, my wife Minerva, my son Renecito, my mom Lupita, and my dad Reynaldo for always believing in me and being so patient with me. Thank you, I love you.

ABSTRACT

Breaking the Weak Commutativity Property of Finite Boolean Subalgebra Triples
(December 2016)

René Montemayor, M.P.S., St. Thomas University, December 2009;

Chair of Committee: Dr. David K. Milovich

Heindorf and Shapiro studied commuting pairs of Boolean subalgebras and showed that Boolean subalgebra pairs that commute retain their commutativity property after taking the exponential functor. Recently Milovich introduced the concept of commuting n-tuples and weakly commuting n-tuples of Boolean subalgebras. In particular, Milovich non-constructively proved that there exists a case where by applying the exponential functor to a triple, of finite Boolean subalgebras, the weakly commuting property is destroyed. The purpose of this paper is to find explicit examples of triples of finite subalgebras that weakly commute but when the exponential functor is applied, the weak commutativity property of this triple is destroyed.

Using *Python*, a high level programming language, it was determined that the four atom finite Boolean Algebra has a triple of subalgebras whose weakly commuting property is destroyed when the exponential functor is applied. Furthermore, we show that the three atom finite Boolean Algebra has no triple of subalgebras whose weakly commuting property is destroyed when the exponential functor is applied.

ACKNOWLEDGEMENTS

I want to thank my wife Minerva for being so patient with me and for bearing with me through so many nights spent doing my homework, reading research articles, and writing this thesis. Without her unwavering support, I would not have been able to finish this Masters program. Thank you for taking care of all the details that helped me to stay focused on finishing this Masters degree. It was truly a work of two, not one and I love you for it.

I also want to thank my advisor Dr. David K. Milovich for his patient guidance throughout this entire endeavor. I want to thank Dr. Rohitha Goonatilake for his unwavering support as one of my committee members as well as the department chair in helping me and encouraging me to finish this Master of Science in Mathematics. I want to thank my other committee members, Dr. Juan H. Hinojosa for his belief in me and for his supportive acceptance of so many of my ideas and Dr. Qingquan Wu for all of his support and advice that went into writing this thesis.

I would like to thank all of my professors, especially Dr. Runchang Lin for his constant support in helping me to overcome so many obstacles that came along before being able to bring this thesis work to a close and Dr. Weam M. Al-Tameemi for never letting me give up when I felt I could not go any further.

I would especially like to thank the Office of Graduate Studies and Research for giving me this opportunity to finish this advanced degree and to make this a reality. Thank you Ms. Virginia Morales for caring so much about all of us graduate students and for going above and beyond in order to help me make it to the end.

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1. Introduction. In his pursuit of making philosophical logic more concrete and quantifiable, Boole [2] established a system of mathematical symbols to represent the logical statements of conjunction, disjunction, and negation called meets (unions), joins (intersections), and complements, respectively. Since this system of representations fits in very naturally to sets—in particular, *fields of sets*—eventually Boole's system of mathematical operations to represent logical statements developed into what we now call the operations on Boolean Algebras (named in honor of Boole) and these Boolean Algebras come in different types and sizes.

A concrete Boolean Algebra is a non-empty set that uses sets as its elements with one element referred to as the bottom set or the empty set and another element referred to as the top set or the universal set. A concrete Boolean Algebra is closed under unions and complements and comes with the following operators: 1) a pair wise intersection for meets ($\wedge$), 2) a pair wise union for joins ($\vee$), and 3) complement with respect to the top set for complements ($-$). We present a more formal definition below following Tao [11].

DEFINITION 1.1 (*Concrete Boolean Algebras*). If X is a set, then a concrete Boolean Algebra or *field of sets* on X is a collection A of subsets of X that obey the following properties:

1) (*Includes the Empty set*) $\emptyset \in A$.

2) (*Closed under Complements*) If $M \in A$ then its complement is also in A, $-M \in A$.

3) (*Closed under Unions*) If $M, N \in A$ then $M \cup N \in A$.

By property 1) and property 2), we claim that the Boolean Algebra A must *include the*

This thesis follows the style of *The Bulletin of Symbolic Logic*.

Universal set X. By property 2) and property 3), we also claim that the Boolean Algebra A is also *closed under intersections*.

PROOF: Let A be a Boolean Algebra made from the subsets of some set X as defined above and let $M, N \in A$. By property 2) $-M, -N \in A$. By property 3) $-M \cup -N \in A$. However, $-M \cup -N = -(M \cap N)$ as can be seen in Figure 1.1 below. Finally, by property 2), if $-(M \cap N) \in A$ then $(M \cap N) \in A$ and we have shown that A is also *closed under intersections* as claimed. □

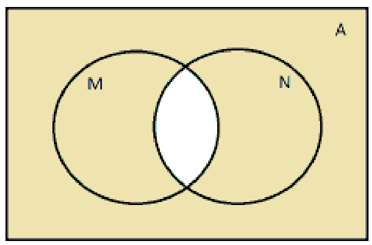

Figure 1.1. Venn Diagram showing the property that $-M \cup -N = -(M \cap N)$.

By Definition 1.1, if A is a collection of subsets of X that satisfies the three axiomatic properties then A is a concrete Boolean Algebra. Now Suppose that A is not just any collection of subsets but in fact a collection of every subset of X, i.e. suppose that A is the power set of X, i.e. $A = \mathcal{P}(X)$. Since every subset of X is in A, then A easily satisfies all three axioms of Definition 1.1 and it should be clear that the power set of X, $\mathcal{P}(X)$, is automatically a concrete Boolean Algebra. We will make use of this fact later and use it as a convenient means to generate concrete Boolean Algebras out of any arbitrary starting universal set. As the reader may now perceive, there must be other kinds of Boolean Algebras which are not restricted to the use of sets at all. We call them abstract Boolean Algebras. Following Givant and Halmos [4] we provide the following definition. For an alternative but equivalent definition, see Koppelberg [3].

DEFINITION 1.2 (*Abstract Boolean Algebras*). An abstract Boolean Algebra is a non-empty set containing two distinguished elements—0 and 1. An abstract Boolean Algebra comes with the following operators: 1) a binary operator for meets ($\wedge$), 2) a binary operator for joins ($\vee$), and 3) a unary operator for complements ($-$) which satisfy the following axioms.

1) $-0 = 1,$ $\quad -1 = 0,$ $\quad -(-p) = p$

2) $p\wedge 0 = 0,$ $\quad p \vee 1 = 1$

3) $p\wedge 1 = p,$ $\quad p \vee 0 = p$

4) $p \wedge -p = 0,$ $\quad p \vee -p = 1$

5) $p\wedge p = p,$ $\quad p \vee p = p$

6) $p \wedge q = q \wedge p,$ $\quad p \vee q = q \vee p$

7) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$ $\quad p \vee (q \vee r) = (p \vee q) \vee r$

8) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$ $\quad p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

9) $-(p\wedge q) = -p \vee -q,$ $\quad -(p \vee q) = -p \wedge -q$

In 1934, Stone [9] published an article showing that every abstract Boolean Algebra is isomorphic to a *field of sets* (a set of prime ideals) which became known as Stone’s representation theorem [9]. The theorem that appeared on that famous publication is presented below. Two key elements in that theorem are ideals and partial orderings ($\leq$) defined below.

DEFINITION 1.3 (*Partial Ordering*). Every Boolean Algebra follows a partial ordering denoted by $\leq$. For a concrete Boolean Algebra, this partial ordering is understood as set inclusion and is the subset operation. For an abstract Boolean Algebra, this partial ordering is understood in terms of meets, joins, and complements. Let A be an abstract

Boolean Algebra. For all $x, y \in A$ we have $x \leq y$ if any of the following equivalent conditions will be met.

1) $x \wedge y = x$

2) $x \vee y = y$

3) $-x \vee y = 1$

4) $x \wedge -y = 0$

DEFINITION 1.4 (*Ideal*). An ideal is a subset of an abstract Boolean Algebra, B, i.e. $I \subset B$ such that

1) $0 \in I$, however $1 \notin I$.

2) For all $a, b \in I$ there exists an element $c \in I$ such that $a, b \leq c$.

3) For all $a \in I$ and for any $b \in B$ if $b \leq a$ then $b \in I$.

DEFINITION 1.5 (*Prime Ideal*). An ideal is a prime ideal, $\mathfrak{T}$, of an abstract Boolean Algebra, B, (or a maximal ideal of a concrete Boolean Algebra) if and only if one of these three equivalent conditions are met.

1) For all $a, b \in B$ if $a \wedge b \in \mathfrak{T}$ then $a \in \mathfrak{T}$ or $b \in \mathfrak{T}$.

2) For all $a \in B$ either $a \in \mathfrak{T}$ or $-a \in \mathfrak{T}$.

3) For any ideal $\mathfrak{J}$ of B, if $\mathfrak{T} \subset \mathfrak{J}$ then $\mathfrak{T} = \mathfrak{J}$. In other words, no ideal of B strictly contains $\mathfrak{T}$.

THEOREM 1.1 (*Stone's Representation Theorem*). Let A be an arbitrary, abstract Boolean Algebra, $\mathfrak{G}$ be the set of all prime ideals in A, $\mathfrak{G}(a)$ be the set of all prime ideals in A which do not contain the element a, and $B(A)$ be the algebra of the sets $\mathfrak{G}(a)$ with join and meet defined as union and intersection, respectively. Then the correspondence $a \leftrightarrow \mathfrak{G}(a)$ defines an isomorphism of A and $B(A)$.

The study of Boolean Algebras received a tremendous boost with Stone's Representation Theorem. In fact, after this discovery, mathematicians from every field of mathematics that use *fields of sets* began exploring Boolean Algebras further and it was soon discovered that Boolean Algebras provided a more convenient way of describing many concepts and calculations in several mathematical fields such as Set Theory, Measure Theory, Topology, Real Analysis, Functional Analysis, and many more.

Note: For the purposes of this work, we shall only consider concrete Boolean Algebras, i.e. fields of sets.

DEFINITION 1.6 (*Subalgebras*). If D is a concrete Boolean Algebra and $A \subset D$, then the set A is called a *subalgebra* of D if A is non-empty and satisfies the axioms of Definition 1.1. Following several examples in the literature, when it is clear, we will denote "A is a subalgebra of D" by $A \leq D$. Otherwise, we will denote "a is a subset of d" by $a \leq d$.

Heindorf and Shapiro [5] studied infinite Boolean Algebras having the projectivity property and the Freese-Nation property. In describing these properties, Heindorf and Shapiro [5] introduced a binary relation between subalgebras which they defined as the commutativity property between subalgebras. Of interest to this work is a version of this relation introduced by Milovich [7] which generalizes this notion to commuting n-tuples of abstract and concrete subalgebras. We place a special focus on commuting triples of concrete (finite) Boolean Algebras. We now present the definition of *commuting subalgebras* introduced by Heindorf and Shapiro [5].

DEFINITION 1.7 (*Commuting Subalgebras*). Given subalgebras $A, B \leq C$ that satisfy Definition 1.3, we say that A and B commute and write $A \rightleftarrows B$ if one of the following equivalent conditions is satisfied for all $a \in A,\ b \in B$.

1) If $a \wedge b = 0$, i.e. a and b are so called *disjoint*, then $a \leq c$ and $b \leq d$, for some disjoint $c, d \in A \cap B$.

2) If $a \leq b$, then $a \leq c \leq b$, for some $c \in A \cap B$.

Milovich [7] generalized this definition into what is called the weakly commuting property of overlapping subalgebra n-tuples. However, in his definition of weakly commuting, Milovich [7] makes use of a different version of Stone's Representation Theorem (Theorem 1.1) which is stated in terms of ultrafilters instead of prime ideals. We now present an alternative version of Stone's Representation theorem based on ultrafilters.

DEFINITION 1.8 (*Filters of Abstract Boolean Algebras*). A *filter*, F, is a subset of an abstract Boolean Algebra, B, i.e. $F \subset B$, if and only if the following three conditions are met.

1) $0 \notin F$, however $1 \in F$.

2) For all $a, b \in F$ there exists an element $c \in F$ such that $a, b \geq c$.

3) For all $a \in F$ and for any $b \in B$ if $b \geq a$ then $b \in F$.

DEFINITION 1.9 (*Ultrafilters of Abstract Boolean Algebras*). A filter is an ultrafilter, $\mathbb{U}$, of an abstract Boolean Algebra, B, (or an ultrafilter of a concrete Boolean Algebra) if and only if one of these three equivalent conditions are met.

1) For all $a, b \in B$ if $a \vee b \in \mathbb{U}$ then $a \in \mathbb{U}$ or $b \in \mathbb{U}$.

2) For all $a \in B$ either $a \in \mathbb{U}$ or $-a \in \mathbb{U}$.

3) For any ultrafilter $\mathbb{V}$ of B, if $\mathbb{U} \subset \mathbb{V}$ then $\mathbb{U} = \mathbb{V}$. In other words, there is no ultrafilter of B strictly containing $\mathbb{U}$.

THEOREM 1.2 (*Stone's Representation Theorem*). Let A be an arbitrary, abstract Boolean Algebra, $\mathfrak{U}$ be the set of all ultrafilters in A, $\mathfrak{U}(a)$ be the set of all ultrafilters in A which contain the element a, and $B(A)$ be the algebra of the sets $\mathfrak{U}(a)$ with join and meet defined as union and intersection, respectively. Then the correspondence $a \leftrightarrow \mathfrak{U}(a)$ defines an isomorphism of A and $B(A)$.

We now provide an alternative definition of commuting subalgebras following Milovich[7].

DEFINITION 1.10 (*Commuting n-tuple Subalgebras*). Given $A_i \leq B$ for $i < n$, we have $\rightleftarrows A$ iff for all $(\mathbb{U}_i)_{i<n} \in \prod_{i<n} \mathfrak{U}(A_i)$, if $\mathbb{U}_i \cap A_j = \mathbb{U}_j \cap A_i$ for all $i, j < n$, then $\bigcup_{i<n} \mathbb{U}_i$ extends to an *ultrafilter* of B.

Below we provide this theorem which was proved by Heindorf and Shapiro [5, p. 98].

THEOREM 1.3 (*Ultrafilter Commuting Property*). Given $A, B \leq C$, A and B will commute, i.e. $A \rightleftarrows B$, if and only if for each ultrafilter $\mathbb{U}$ of A and each ultrafilter $\mathbb{V}$ of B, if $\mathbb{U} \cap B = \mathbb{V} \cap A$, then $\mathbb{U} \cup \mathbb{V}$ extends to an ultrafilter of C.

In order to experiment with Heindorf and Shapiro's [5] commuting subalgebra property, we use a convenient means of transforming an ordinary, concrete Boolean Algebra into a concrete Boolean Algebra made up of filters and sets of filters. We do this by using a functor called the exponential (exp) which is the Stone Dual of the Vietoris Hyperspace functor (Exp). Bankston [1, p.300] explained that there exists a Stone duality between the closed nonempty subsets of a Boolean space and the proper filters of its corresponding Boolean Algebra. The Vietoris Hyperspace was introduced in 1922 by Leopold Vietoris [12] as a generalization of the Hausdorff metric and is a topological construction on compact Hausdorff spaces. This functor and other topological functors have been studied in much detail by authors such as Ščepin [10]. Therefore we have several alternatives for the exponential functor. We have the filter construction of the exponential by Milovich [7], the algebraic construction of the exponential by Bankston [1], and the sets-of-functions construction of the exponential by Heindorf and Shapiro [5]. By Stone's Representation Theorem, we are able to find a dual (or Stone dual) of this topological functor (Exp) that works in a similar fashion using Boolean Algebras (exp). For the purposes of this work, we will only be interested in the exponential functor (exp) acting on concrete Boolean Algebras.

DEFINITION 1.11 (*Functors*). For any two mathematical categories A and B, a *functor* F is a mapping that associates each object x in A to some object $F(x)$ in B, and associates each morphism mapping in A, $f: x \to y$, to a morphism mapping in B, $F(f): F(x) \to F(y)$, such that identity morphisms and composition morphisms are preserved, i.e. $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$ and $F(g \circ f) = F(f) \circ F(g)$ for all morphisms $f: x \to y$ and $g: y \to z$ in A.

Following Milovich [7, p.49] we now provide a definition of the exponential functor. For an alternative but equivalent definition of the exponential functor, see Heindorf and Shapiro [5, p. 78].

DEFINITION 1.12 (*Exponential Functor, exp*). Let *Bool* be the mathematical category of abstract Boolean Algebras and Boolean homomorphisms. The exponential, exp, is a functor from *Bool* to *Bool* defined as follows.

1) $\mathrm{Filt}(A) = \{F \subset A | F \text{ is a (proper) filter of A}\}$

2) $[a]_A = \{F \in \text{Filt}(A) | a \in F\}$

3) $\exp(A) = \langle\{[a]_A | a \in A\}\rangle \leq \mathcal{P}(\text{Filt}(A))$, where $\langle\{[a]_A | a \in A\}\rangle$ is the subalgebra generated by $\{[a]_A | a \in A\}$.

4) $\exp(f: A \to B)\,([a]_A) = [f(a)]_B$

Following Mendelson [6, p.129], we provide the following definition of a principal filter.

DEFINITION 1.13 (*Principal Filters*). Let F be a filter of an abstract Boolean Algebra (alternatively a concrete Boolean Algebra), B. F is a *principal filter* if it has a minimum element.

We can now say that if A is a finite Boolean Algebra, then $\exp(A) = \mathcal{P}(\text{Filt}(A))$ because every filter of A is a principal filter.

We now present a generalized version of the commuting property of subalgebras following Milovich [7].

DEFINITION 1.14 (*Boolean Embeddings are Boolean Homomorphisms*). A Boolean embedding is an injective Boolean homomorphism.

DEFINITION 1.15 (*Identity Mapping*). Natural identity mappings denoted by id are well-defined Boolean homomorphisms. If B is an abstract Boolean Algebra with subalgebra A, $A \leq B$, then $\text{id}: A \to B$ such that for all $a \in A$, $\text{id}(a) = a$.

The exponential of the identity mapping is a Boolean embedding and an isomorphism onto its range. Applying Definition 1.12, whenever $A \leq B$ we have $\exp(\text{id}: A \to B): \exp(A) \to \exp(B)$ and we let

$$\exp_{\text{B}}(A) = \text{ran}(\exp(id: A \to B)).$$

Then observe that if $A_i \leq B$ for $i < n$, then

$$\mathrm{ran}(\exp\ (\mathrm{id}\colon \bigcap A \to B)) = \bigcap_{i<n} \mathrm{ran}(\exp(\mathrm{id}\colon A \to B)).$$

DEFINITION 1.16 (*Weakly Commuting n-tuple Subalgebras*). Given $A_i \leq B$ for $i < n$, we say that A_i weakly commutes and write $\rightleftharpoons A_i$ if, for all $(x_i)_{i<n} \in \prod_{i<n} A_i$ satisfying $\bigwedge_{i<n} x_i = 0$, then there exists $(y_i)_{i<n} \in \prod_{i<n} \langle A_i \cap \bigcup_{j \neq i} A_j \rangle$ such that $\bigwedge_{i<n} y_i = 0$ and for all $i < n$, $x_i \leq y_i$. Given A_i, x_i, and y_i as above, we call any y_i a weak incompatibility witness for x_i. Where for $n = 2$, this weakly commuting property of n-tuple subalgebras is identically the commuting property of subalgebra pairs as characterized by Heindorf and Shapiro [5].

DEFINITION 1.17 (*Weakly Commuting Well*). Given $A_i \leq B$ for $i < n$, we say that A_i (weakly) commutes well if $(A_i)_{i<m}$ (weakly) commutes for each $m \leq n$.

We are now ready to state the following theorems and corollary by Milovich [7] that show that the exponential functor destroys the commuting property of a subalgebra triple.

THEOREM 1.4 (*Commutes Well*). If $(A_i)_{i<n}$ weakly commutes well, then it commutes well.

THEOREM 1.5 (*Destroying the Weak Commuting Property in Subalgebra Triples*). There exists a finite Boolean Algebra B with subalgebras A_i for $i < 3$ such that $(A_i)_{i<3}$ commutes well in B but $(\mathrm{ran}(\exp(\mathrm{id}\colon A_i \to B)))_{i<3}$ does not weakly commute well in $\exp(B)$.

COROLLARY 1.6 (*Destroying the Commuting Property in Subalgebra Triples*). For some $n \geq 2$, there exists a finite Boolean Algebra B with subalgebras A_i for $i < 3$ such that $(A_i)_{i<3}$ commutes in B but $(\mathrm{ran}(\exp(\mathrm{id}\colon A_i \to B)))_{i<3}$ does not weakly commute in $\exp(B)$.

Alongside this Theorem 1.5 and Corollary 1.6, Milovich [7] presented the problem below to ask for a specific example of Theorem 1.5. It is the motivation for this work, to find a solution to this problem presented below.

PROBLEM 1.1 Find small Boolean Algebras $A_0, A_1, A_2 \leq B$ witnessing the above theorem for $F = \exp$.

Since the focus of Problem 1.1 is finding three subalgebras that weakly commute but then fail to weakly commute after the exponential is applied, we will refer to this weakly commuting (triple) property as three-way commuting and offer the following definition which is a more explicit formulation of Definition 1.16 for the case $n = 3$.

DEFINITION 1.18 (*Three-Way Commuting*). Let A, B, and C be three subalgebras of D, a concrete Boolean Algebra. If for all $(a, b, c) \in A$ x B x C such that $a \wedge b \wedge c = 0$, there exists an $(x, y, z) \in \langle A \cap (B \cup C)\rangle$ x $\langle B \cap (A \cup C)\rangle$ x $\langle C \cap (A \cup B)\rangle$ such that if

$x \wedge y \wedge z = 0$ and $a \leq x$, $b \leq y$, and $c \leq z$ then A, B, and C three-way commute in D and we write $A_D \trianglelefteq B_D \trianglelefteq C_D$. If A, B, and C fail to three-way commute in D, we will denote it by $A_D \ntrianglelefteq B_D \ntrianglelefteq C_D$.

If $A_D \trianglelefteq B_D \trianglelefteq C_D$ and $\exp_D(A) \ntrianglelefteq \exp_D(B) \ntrianglelefteq \exp_D(C)$ then we will say that $A, B, C \leq D$ witnesses Theorem 1.5. Following Milovich [7], we now provide a very useful theorem for meeting the objective of this work.

THEOREM 1.6 (*Unlimited Inheritance of Weak Commuting Subalgebra Triples*). If A is a concrete Boolean Algebra with three subalgebras, $A_0, A_1, A_2 \leq A$, witnessing Theorem 1.5, then every concrete Boolean Algebra with an embedding from A will also admit a witness to Theorem 1.5.

PROOF: Let D be a concrete Boolean Algebra that witnesses the three-way commuting property but fails to three-way commute once the exponential functor is applied to the three-way commuting subalgebras, i.e. for $A, B, C \leq D$, $A_D \trianglelefteq B_D \trianglelefteq C_D$ but $\exp_D(A) \ntrianglelefteq \exp_D(B) \ntrianglelefteq \exp_D(C)$. Let D' be a concrete Boolean Algebra that contains D, i.e. has D as an embedding, $D' \supseteq D$. Since $A, B, C \leq D$ implies that $A, B, C \leq D'$ then D' has *inherited* the same three-way commuting properties by virtue of the embeddings of A, B, and C into D'. Since D' was chosen arbitrarily, then any concrete Boolean Algebra having D as an embedding will also *inherit* the same three subalgebras that three-way commute in D yet fail to three-way commute after the exponential functor has been applied. □

We now provide another tool for studying the three-way commuting property. We will take advantage of the atomic character of finite concrete Boolean Algebras in order to generalize our findings. We provide the following definition following Solomon [8].

DEFINITION 1.19 (*Atoms*). Let A be an abstract Boolean Algebra. If $x \in A$ and $x = 0$, then x is an atom of A if for every $y \in A$, if $y \leq x$ then $y = x$ or $y = 0$. Analogously for some concrete Boolean Algebra, C, if $a \in C$ and a is an atom of C, then for every $y \in C$ if $y \leq a$ then $y = a$ or $y = \emptyset$. It should be clear to the reader that under these circumstances every element of a finite, concrete Boolean Algebra is a finite union of atoms.

DEFINITION 1.20 (*Generators of Principal Filters*). If F is a principal filter and $a \in F$ is the least element in F, then a is a *generator of the principal filter* F. Since a with all of its upper bounds makes up the principal filter, F, we will use the following notation, $[a]^{\uparrow}$, to denote F, i.e. $F = [a]^{\uparrow}$.

The number of elements generated by the exponential functor are expected to be too many to manage a check of the 3-Way Commuting property, therefore we will need a more manageable way to check the 3-Way Commuting property. For that purpose, we provide the following lemma proposed by Milovich [7].

LEMMA 1.1 (*Atom-wise Commuting*). In order to show the 3-Way Commuting property of a subalgebra triple, given any atom triple $(x_i)_{i<3}$ of $(A_i)_{i<3} \leq A$ such that $\bigwedge_{i<3} x_i = 0$, it suffices to find atoms $(y_i)_{i<3}$ as in Definition 1.16.

Before presenting one more useful theorem for solving this problem, we first provide the following Theorem proven by Solomon [8].

THEOREM 1.7 (*Isomorphism of Equivalent, Concrete Boolean Algebras*). Finite Boolean Algebras with equal numbers of elements are isomorphic. Equivalently, all finite, concrete Boolean Algebras sharing the same cardinality or the same *number of atoms* are isomorphic.

THEOREM 1.8 (*Three-Way Commuting is Preserved Atom-wise*). If A is a finite concrete Boolean Algebra having n atoms that admits a witness of Theorem 1.5, then every concrete Boolean Algebra, B, having $a \geq n$ atoms (the same atomic number or higher) will also admit a witness of Theorem 1.5.

Theorem 1.6 together with Theorem 1.7 gives a straight forward proof of Theorem 1.8.

We are now ready to provide an equivalent version of Theorem 1.5 using atomic references instead of embeddings.

We provide an observation made by Milovich [7, p. 17] then finish this section with an *atomic* rephrasing of Problem 1.1

PROBLEM 1.2 What is the atomic number of the smallest, atomic, concrete Boolean Algebra that has the three-way commuting property destroyed by the exponential functor?

2. Method. In order to find a solution to this problem we decided to automate the process. We used a program language called Python and encoded the instructions for generation of subalgebras given a preset number of generators that operated within the limits of a preset universal set. We then automated the checking of the commuting property of subalgebra triples. Finally, we encoded the instructions for forming the exponentials of each of the subalgebra triples and then checked them for the commuting property to confirm whether or not the commuting property of the subalgebra triples was indeed destroyed by the exponential functor. The Python code used is explained below. The reader should know up front that Python does not use numbered lines as part of its syntax. The numbering added to the Python code below is purely for editorial reasons in order to make the program code easier to follow.

```
1    f=frozenset
2    def fr(lis):
3            return set([frozenset(x) for x in lis])
```

The command "*frozenset*" was used because the Python "*set*" data type only admits constants as elements. So, in order to manipulate the Boolean Algebras expressed as fields of sets, we make use of the "*frozenset*" data type which are admissible inner sets or sets of sets.

```
4    universe=f(range(1,5))
5    genA=fr([[2,3],[1]])
6    genB=fr([[1,2],[3]])
7    genC=fr([[3,4],[2]])
```

The parent Boolean Algebra, D, is generated from the top set called "*universe*" which is restricted to having four atoms namely $\{1,2,3,4\}$. The three subalgebras of the parent Boolean Algebra, A, B, and C are generated by $\langle\{2,3\},\{1\}\rangle$, $\langle\{1,2\},\{3\}\rangle$, and $\langle\{3,4\},\{2\}\rangle$ respectively.

```
def comps(g, top):
return g|set([top-x for x in g])
```

The submodule "*comps*" is used to generate closure under complements.

```
def meets(s):
m=s
for x in s:
m=m|set([x&y for y in m])
return(m)
```

The submodule "*meets*" is used to generate closure under intersections.

```
def joins(s):
j=s
for x in s:
j=j|set([x|y for y in j])
return (j)
```

The submodule "*joins*" is used to generate closure under unions.

```
def subalg(g,top):
return (joins(meets(comps(g,top))))
```

The submodule "*subalg(g,top)*" generates subalgebras based on a list of generators and the top set we called "*universe*".

```
def genOverlap(algA,algB,algC,top):
genO=algA&(algB|algC)
return genO
```

Once the subalgebras have been generated, three overlapping subalgebras that satisfy Definition 1.18 need to get generated as well. The first step is to pick the generators that we will use to generate each of these overlapping subalgebras and this is accomplished with the submodule “*genOverlap(algA,algB,algC,top)*”.

```
def overlapingalgs(algA,algB,algC,top):
O=subalg(algA&(algB|algC),top)
return O
```

Once the generators for the overlapping subalgebras have been picked, the submodule “*overlapingalgs(algA,algB,algC,top)*” is used repeatedly with different generators to generate each of the three overlapping algebras called P, Q, and R.

```
def atom(A):
atoms=set()
for x in A:
addx=False
if not x==frozenset():
addx=True
for y in atoms:
    if y<=x:
        addx=False
        break
```

```
if addx==True:
atoms=set([y for y in atoms if not x<=y])
atoms.add(x)
return(atoms)
```

The submodule “*atom(A)*” is used to pick out the atoms of any subalgebra.

```
def cand(algA,algB,algC):
return([[x,y,z] for x in algA for y in algB for z in algC if x&y&z==frozenset()])
```

The submodule “*cand(algA,algB,algC)*” is used to select each candidate triple satisfying the conditions of Definition 1.18, $(a, b, c) \in A$ x B x C such that $a \wedge b \wedge c = 0$.

```
def tway(abc,xyz):
for x,y,z in abc:
found=False
for u,v,w in xyz:
if x<=u and y<=v and z<=w:
        found=True
        break
if found==False:
        break
return("True") if found==True else ("False for abc = ", (x,y,z))
```

The submodule “*tway(abc,xyz)*” implements Definition 1.18 by taking a triplet from the three subalgebras A, B, and C and another triplet from the overlapping subalgebras P, Q, R and then verifies if these have the 3-way commuting property.

```
def filters(A):
```

```
return (f(fr([[y for y in A if y>=x]for x in A if not x==frozenset()])))
```

The submodule “*filters(A)*” picks out the filters of any subalgebra.

```
def xgen(alg, allfilters):
li=[[f for f in allfilters if x in f]for x in alg]
return fr(li)
```

The submodule “*xgen(alg, allfilters)*” picks the generators that will be needed to generate the exponential functor of each subalgebra.

```
def xalg(alg,allfilters):
x=xgen(alg,allfilters)
return subalg(x,allfilters)
```

The submodule “*def xalg(alg,allfilters)*” generates the exponential of any subalgebra.

```
genD=fr([[x] for x in universe])
algD=subalg(genD,universe)
```

The command “*genD*” is used to pick the generators which will be used to generate the parent Boolean Algebra, D using the submodule “*subalg(genD,universe)*”.

```
A=subalg(genA,universe)
B=subalg(genB,universe)
C=subalg(genC,universe)
```

Each of the three subalgebras, $A, B, C \leq D$, are generated using the “*subalg()*” module.

```
genP=genOverlap(A,B,C,universe)
genQ=genOverlap(B,A,C,universe)
genR=genOverlap(C,B,A,universe)
```

The generators that will be used to generate the three overlapping subalgebras, $P, Q, R \leq D$, are picked using the "*genOverlap()*" submodule.

```
P=overlapingalgs(A,B,C,universe)
Q=overlapingalgs(B,A,C,universe)
R=overlapingalgs(C,B,A,universe)
```

Each of the three overlapping subalgebras, $P, Q, R \leq D$, are generated using the "*overlapingalgs()*" module where $P = \langle A \cap (B \cup C) \rangle$, $Q = \langle B \cap (A \cup C) \rangle$, and $R = \langle C \cap (A \cup B) \rangle$ satisfying Definition 1.18. We then write the outputs to a text file for post-retrieval and post-analysis after running the Python code. The number of text files and the type of information that can be written to them is unlimited. In the least, we would want to know what the subalgebras that witness the 3-way commuting property look like, what triples are used to test the 3-way commuting property, and how the 3-way commuting property is destroyed by the exponential functor.

```
eFile=open("RESULT_algebras.txt", "w")
eFile.write('\n RENESAYS \n'.join(["Parent algebra D is", str([[x] for x in algD]),
    "subalgebra A is", str([[x] for x in A]), "subalgebra B is", str([[x] for x in B]),
    "subalgebra C is", str([[x] for x in C]),""]))
eFile.write('\n RENESAYS \n'.join(["subalgebra P is", str([[x] for x in P]),
    "subalgebra Q is", str([[x] for x in Q]), "subalgebra R is", str([[x] for x in R]),""]))
eFile.close()
```

The Python command "*eFile=open("RESULT_algebras.txt", "w")*" with the attribute "*w*" is used to initialize the text file called "*RESULT_algebras.txt*". The Python commands *eFile=open()*, *eFile.write()*, and *eFile.close()* are always used together in order to control the opening, writing, and closing of the text file used to save the desired output.

```
aD=atom(algD)
aA=atom(A)
aB=atom(B)
aC=atom(C)
aP=atom(P)
aQ=atom(Q)
aR=atom(R)
```

The submodule “*atom()*” is used to pick the atoms of each of the 6 subalgebras, $A, B, C, P, Q, R \leq D$.

```
eFile=open("RESULT_atoms_algebras.txt", "w")
eFile.write('\n RENESAYS \n'.join(["Atoms of Parent D are", str([[x] for x in aD]),
    "Atoms of A are", str([[x] for x in aA]), "Atoms of B are", str([[x] for x in aB]),
    "Atoms of C are", str([[x] for x in aC]),"Atoms of P are", str([[x] for x in aP]),
    "Atoms of Q are", str([[x] for x in aQ]), "Atoms of R are", str([[x] for x in aR]),""]))
eFile.close()
```

We save the atoms of each of the subalgebras outputted by the Python program for later analysis.

```
abc=cand(aA,aB,aC)
xyz=cand(aP,aQ,aR)
```

The submodule “*cand()*” is used to pick out every candidate triple that satisfies Definition 1.18.

```
eFile=open("RESULT_candidates_algebras.txt", "w")
eFile.write('\n RENESAYS \n'.join(["abc Candidates are", str(abc), "xyz Candidates are", str(xyz),""]))
eFile.close()
```

We save the candidate triplets for later analysis to the text file called "*RESULT_candidates_algebras.txt*".

```
result=tway(abc,xyz)
```

The submodule "*tway()*" takes each candidate triple and checks the 3-way commuting property.

```
eFile=open("RESULT_tcommute_algebras.txt", "w")
eFile.write('\n RENESAYS \n'.join(["abc/xyz t-commute Results are", str(result), ""]))
eFile.close()
```

We save the results of the 3-way commuting property check to the text file "*RESULT_tcommute_algebras.txt*" for later analysis.

```
fD=filters(algD)
```

We pick the filters of the parent Boolean Algebra D.

```
genxA=xgen(A,fD)
genxB=xgen(B,fD)
genxC=xgen(C,fD)
```

We pick the generators that will be used to generate the three exponential subalgebras $\exp_D(A), \exp_D(B), \exp_D(C) \leq \exp(D)$.

```
xA=xalg(A,fD)
```

```
xB=xalg(B,fD)
xC=xalg(C,fD)
```

We now generate the exponential subalgebras $xA = \exp_D(A)$, $xB = \exp_D(B)$, $xC = \exp_D(C) \leq \exp(D)$.

```
genxP=genOverlap(xA,xB,xC,fD)
genxQ=genOverlap(xB,xA,xC,fD)
genxR=genOverlap(xC,xB,xA,fD)
```

We pick the generators that will be used to generate the three overlapping exponential subalgebras.

```
xP=overlapingalgs(xA,xB,xC,fD)
xQ=overlapingalgs(xB,xA,xC,fD)
xR=overlapingalgs(xC,xB,xA,fD)
```

We now generate the three corresponding exponential subalgebras $xP = \langle xA \cap (xB \cup xC)\rangle$, $xQ = \langle xB \cap (xA \cup xC)\rangle$, and $xR = \langle xC \cap (xA \cup xB)\rangle$.

```
axA,axB,axC =atom(xA), atom(xB), atom(xC)
axP,axQ,axR =atom(xP), atom(xQ), atom(xR)
```

We use the "*atom()*" submodule to pick the atoms of the exponential subalgebras.

```
eFile=open("RESULT_atoms_exp_algebras.txt", "w")
eFile.write('\n RENESAYS \n'.join(["Atoms of exponential subalgebra A is", str([[x]
    for x in axA]), "Atoms of exponential subalgebra B is", str([[x] for x in axB]),
    "Atoms of exponential subalgebra C is", str([[x] for x in axC]),"Atoms of
    exponential subalgebra P is", str([[x] for x in axP]), "Atoms of exponential
```

```
subalgebra Q is", str([[x] for x in axQ]), "Atoms of exponential subalgebra R is",
str([[x] for x in axR]),""]))
eFile.close()
```

We save the atoms of the exponential subalgebras to the file called "*RESULT_atoms_exp_algebras.txt*" for later analysis.

```
Eabc=cand(axA,axB,axC)
Exyz=cand(axP,axQ,axR)
```

The submodule "*cand()*" picks every candidate triple from the exponential subalgebras that satisfies Definition 1.18.

```
eFile=open("RESULT_candidates_exp_algebras.txt", "w")
eFile.write('\n RENESAYS \n'.join(["Eabc Candidates are", str(Eabc), "Exyz
Candidates are", str(Exyz),""]))
eFile.close()
```

We save the candidate triples of the exponential subalgebras to the file called "*RESULT_candidates_exp_algebras.txt*" for later analysis.

```
count=0
tway_count=0
for (x,y,z) in Eabc:
Eresult=tway([(x,y,z)],Exyz)
tway_count=tway_count+1
if Eresult!="True":
found=Eresult[0]
(x,y,z)=Eresult[1]
```

```
count=count+1
```

We are now ready to find a witness of Theorem 1.5. We check the 3-way commuting property in the exponential subalgebras. We carefully track this check so that we can identify a specific example where the 3-way commuting property is destroyed by the exponential functor.

```
eFile=open("RESULT_tcommute_exp_algebras.txt", "w")
aFile=open("RESULT_tcommute_atoms_exp_algebras.txt", "w")
eFile=open("RESULT_tcommute_exp_algebras.txt", "a")
aFile=open("RESULT_tcommute_atoms_exp_algebras.txt", "a")
eFile.write('\n RENESAYS \n'.join(["Eabc/Exyz t-commute Results are", found,
    "RENESAYS with x,y,z #", str(count), str([[f] for f in (x,y,z)]), ""]))
eFile.close()
aFile.write('\n RENESAYS \n'.join(["Eabc/Exyz t-commute Results are", found,
    "RENESAYS with x,y,z #", str(count), str([[atom(f) for f in fs] for fs in (x,y,z)]), ""]))
aFile.close()
eFile=open("RESULT_tcommute_exp_algebras.txt", "a")
eFile.write('\n RENESAYS \n'.join(["Eabc/Exyz t-commute Results are",
str(Eresult), "RENESAYS for all x,y,z candidates #", str(tway_count), ""]))
eFile.close()
```

We save the results of the 3-way commuting property check to two different text files. One records just the atoms of the candidate triples of the exponential subalgebra, "*RESULT_tcommute_atoms_exp_algebras.txt*". The other records the full candidate triples of the exponential subalgebras, "*RESULT_tcommute_exp_algebras.txt*". We use the attribute "*a*" in order to append the output to the text file as it is getting generated without deleting any prior output results that were saved earlier.

```
print("DONE")
```

We end the Python program with the visual cue "*DONE*" just as a visual verification of the Python program running through all of its code successfully until the end.

3. Results. We investigated the weak (three-way) commuting property of the three atom Boolean Algebra. We were unable to find three subalgebras of the three atom Boolean Algebra that possessed the weak (three-way) commuting property even before applying the exponential functor. The results of this three Boolean Algebra investigation are provided in Appendix A.

The smallest Boolean Algebra that witnesses Theorem 1.5 is the four atom Boolean Algebra shown below.

$$D = \left\{ \begin{matrix} \{1,2,3,4\}, \\ \{1\},\{2\},\{3\},\{4\}, \\ \{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}, \\ \{1,2,4\},\{2,3,4\},\{1,3,4\},\{1,2,3\}, \\ \{\quad\} \end{matrix} \right\}$$

Three subalgebras of D, $A, B, C \leq D$, were generated as shown below.

$$A = \langle \{2,3\},\{1\} \rangle = \left\{ \begin{matrix} \{1,2,3,4\}, \\ \{1\},\{4\},\{1,4\},\{2,3\}, \\ \{2,3,4\},\{1,2,3\}, \\ \{\quad\} \end{matrix} \right\}$$

$$B = \langle \{1,2\},\{3\} \rangle = \left\{ \begin{matrix} \{1,2,3,4\}, \\ \{3\},\{4\},\{1,2\},\{3,4\}, \\ \{1,2,4\},\{1,2,3\} \\ \{\quad\} \end{matrix} \right\}$$

$$C = \langle \{3,4\},\{2\} \rangle = \left\{ \begin{matrix} \{1,2,3,4\}, \\ \{1\},\{2\},\{1,2\},\{3,4\}, \\ \{2,3,4\},\{1,3,4\}, \\ \{\quad\} \end{matrix} \right\}$$

Three overlapping subalgebras, $P, Q, R \leq D$, satisfying Definition 1.18 were generated as follows.

$$P = \langle A \cap (B \cup C) \rangle = \langle \{1,2,3,4\},\{4\},\{1\},\{1,2,3\},\{2,3,4\},\{\,\} \rangle$$

$$= \left\{ \begin{matrix} \{1,2,3,4\}, \\ \{1\},\{4\},\{2,3\},\{1,4\}, \\ \{1,2,3\},\{2,3,4\}, \\ \{\quad\} \end{matrix} \right\}$$

$$Q = \langle B \cap (A \cup C) \rangle = \langle \{1,2,3,4\}, \{4\}, \{1,2\}, \{3,4\}, \{1,2,3\}, \{\ \} \rangle$$

$$= \left\{ \begin{array}{c} \{1,2,3,4\}, \\ \{3\}, \{4\}, \{1,2\}, \{3,4\}, \\ \{1,2,4\}, \{1,2,3\}, \\ \{\quad\} \end{array} \right\}$$

$$R = \langle C \cap (A \cup B) \rangle = \langle \{1,2,3,4\}, \{1\}, \{1,2\}, \{3,4\}, \{2,3,4\}, \{\ \} \rangle$$

$$= \left\{ \begin{array}{c} \{1,2,3,4\}, \\ \{1\}, \{2\}, \{1,2\}, \{3,4\}, \\ \{1,3,4\}, \{2,3,4\}, \\ \{\quad\} \end{array} \right\}$$

We observe that $A = P$, $B = Q$, and $C = R$, so therefore any triple $(a, b, c) \in A$ x B x C such that $a \wedge b \wedge c = 0$ has a corresponding triple $(x, y, z) \in P$ x Q x R such that $x \wedge y \wedge z = 0$ and $a \leq x,\ b \leq y,\ c \leq z$. In this way we have verified the 3-way commuting property, $A_D \trianglelefteq B_D \trianglelefteq C_D$. For the curious reader, we list every triple in Appendix B for further verification.

To show that the exponential functor destroys the 3-way commute property, we quickly realize how difficult and tedious it will be to list the exponential of each subalgebra, a set of sets of sets of filters of each subalgebra. The cardinality of the exponential of subalgebra A is too big, $n(\exp_D(A)) = 128$. Furthermore, each element of $\exp_D(A)$ is a set of sets of filters from A. Therefore, we make use of Theorem 1.8 and Lemma 1.1 in order to show the 3-way commuting property using just the atoms of the respective subalgebras.

Below we list the atoms of each of the subalgebras from the prior section above after the exponential functor was applied. We list each exponential subalgebra by means of Definition 1.20 by using the generators of the principal filters. A complete list of the atoms of the exponential subalgebras and their overlapping subalgebras are listed in Appendix C.

$$\exp_{\mathrm{D}}(A)_{\mathrm{atoms}} = \left\{\begin{array}{c} [\{4\}]^{\uparrow}, \\ \{[\{1,2,3\}]^{\uparrow}, [\ \{1,2\}]^{\uparrow}, [\{1,3\}]^{\uparrow}\}, \\ [\{1,4\}]^{\uparrow}, \\ \{[\{1,2,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow}, \{1,2,3,4\}\} \\ \{[\{2\}]^{\uparrow}, [\{3\}]^{\uparrow}, [\{2,3\}]^{\uparrow}\}, \\ \{[\{2,4\}]^{\uparrow}, [\{2,3,4\}]^{\uparrow}, [\{3,4\}]^{\uparrow}\}, \\ [\{1\}]^{\uparrow} \end{array}\right\}$$

$$\exp_{D}(B)_{\mathrm{atoms}} = \left\{\begin{array}{c} [\{3,4\}]^{\uparrow}, \\ \{[\{1,4\}]^{\uparrow}, [\{1,2,4\}]^{\uparrow}, [\{2,4\}]^{\uparrow}\}, \\ [\{4\}]^{\uparrow}, \\ \{[\{1,3\}]^{\uparrow}, [\{1,2,3\}]^{\uparrow}, [\{2,3\}]^{\uparrow}\}, \\ \{[\{1\}]^{\uparrow}, [\{2\}]^{\uparrow}, [\{1,2\}]^{\uparrow}\}, \\ \{[\{2,3,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow}, \{1,2,3,4\}\}, \\ [\{3\}]^{\uparrow} \end{array}\right\}$$

$$\exp_D(C)_{\text{atoms}} = \left\{\begin{array}{c} [\{1,2\}]^{\uparrow}, \\ \{[\{1,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow}, [\{1,3\}]^{\uparrow}\}, \\ \{[\{4\}]^{\uparrow}, [\{3,4\}]^{\uparrow}, [\{3\}]^{\uparrow}\}, \\ \{[\{1,2,4\}]^{\uparrow}, [\{1,2,3\}]^{\uparrow}, \{1,2,3,4\}\}, \\ \{[\{2,3,4\}]^{\uparrow}, [\{2,4\}]^{\uparrow}, [\{2,3\}]^{\uparrow}\}, \\ [\{1\}]^{\uparrow}, \\ [\{2\}]^{\uparrow} \end{array}\right\}$$

The three overlapping exponential subalgebras satisfying Definition 1.18 were generated as follows.

$$\langle \exp_D(A) \cap [\exp_D(B) \cup \exp_D(C)] \rangle = \left\{\begin{array}{c} \left\{\begin{array}{c} [\{1,4\}]^{\uparrow}, [\{1,2,4\}]^{\uparrow}, \\ [\{1,3,4\}]^{\uparrow}, \{1,2,3,4\} \end{array}\right\}, \\ [\{4\}]^{\uparrow}, \\ \{[\{1,2,3\}]^{\uparrow}, [\{1,2\}]^{\uparrow}, [\{1,3\}]^{\uparrow}\}, \\ \{[\{2\}]^{\uparrow}, [\{3\}]^{\uparrow}, [\{2,3\}]^{\uparrow}\}, \\ \{[\{2,3,4\}]^{\uparrow}, [\{3,4\}]^{\uparrow}, [\{2,4\}]^{\uparrow}\}, \\ [\{1\}]^{\uparrow} \end{array}\right\}$$

$$\langle \exp_D(B) \cap [\exp_D(A) \cup \exp_D(C)] \rangle = \left\{ \begin{array}{c} [\{3,4\}]^{\uparrow}, \\ [\{4\}]^{\uparrow}, \\ \{[\{1,2,3\}]^{\uparrow}, [\{1,3\}]^{\uparrow}, [\{2,3\}]^{\uparrow}\}, \\ \left\{ \begin{array}{c} [\{2,4\}]^{\uparrow}, [\{1,4\}]^{\uparrow}, \{1,2,3,4\}, \\ [\{1,2,4\}]^{\uparrow}, [\{2,3,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow} \end{array} \right\}, \\ \{[\{1\}]^{\uparrow}, [\{2\}]^{\uparrow}, [\{1,2\}]^{\uparrow}\}, \\ [\{3\}]^{\uparrow} \end{array} \right\}$$

$$\langle \exp_D(C) \cap [\exp_D(A) \cup \exp_D(B)] \rangle = \left\{ \begin{array}{c} [\{1,2\}]^{\uparrow}, \\ \{[\{4\}]^{\uparrow}, [\{3,4\}]^{\uparrow}[\{3\}]^{\uparrow}\}, \\ \{[\{2,3,4\}]^{\uparrow}, [\{2,4\}]^{\uparrow}, [\{2,3\}]^{\uparrow}\}, \\ [\{1\}]^{\uparrow}, \\ \left\{ \begin{array}{c} [\{1,4\}]^{\uparrow}, \{1,2,3,4\}, [\{1,3\}]^{\uparrow}, \\ [\{1,2,4\}]^{\uparrow}, [\{1,2,3\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow} \end{array} \right\}, \\ [\{2\}]^{\uparrow} \end{array} \right\}$$

The three-way commuting property fails for the exponential of the subalgebras. Of the 328 triples $(a, b, c) \in \exp_D(A) \text{ X } \exp_D(B) \text{ X } \exp_D(C)$ such that $a \wedge b \wedge c = 0$ there are four (a, b, c) triples that have no corresponding triple of the potential 206 triples $(x, y, z) \in \langle \exp_D(A) \cap [\exp_D(B) \cup \exp_D(C)] \rangle \text{ X } \langle \exp_D(B) \cap [\exp_D(A) \cup \exp_D(C)] \rangle \text{ X } \langle \exp_D(C) \cap [\exp_D(A) \cup \exp_D(B)] \rangle$ such that $x \wedge y \wedge z = 0$ to satisfy the condition that $a \leq x,\ b \leq y,\ c \leq z$. The four triples that prove that the three-way commuting property was destroyed by the exponential functor are listed below.

$$(a,b,c)_{1,\mathrm{exp}} = \begin{pmatrix} [\{1,4\}]^{\uparrow}, \\ \left\{\{[\{1,4\}]^{\uparrow}, [\{1,2,4\}]^{\uparrow}, [\{2,4\}]^{\uparrow}\}\right\}, \\ \left\{[\{1,2,4\}]^{\uparrow}, [\{1,2,3\}]^{\uparrow}, \{1,2,3,4\}\right\} \end{pmatrix}$$

$$(a,b,c)_{2,\mathrm{exp}} = \begin{pmatrix} [\{1,4\}]^{\uparrow}, \\ \left\{[\{2,3,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow}, \{1,2,3,4\}\right\}, \\ \left\{[\{1,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow}, [\{1,3\}]^{\uparrow}\right\} \end{pmatrix}$$

$$(a,b,c)_{3,\mathrm{exp}} = \begin{pmatrix} [\{1,4\}]^{\uparrow}, \\ \left\{[\{2,3,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow}, \{1,2,3,4\}\right\}, \\ \left\{[\{1,2,4\}]^{\uparrow}, [\{1,2,3\}]^{\uparrow}, \{1,2,3,4\}\right\} \end{pmatrix}$$

$$(a,b,c)_{4,\mathrm{exp}} = \begin{pmatrix} \left\{[\{1,2,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow}, \{1,2,3,4\}\right\}, \\ \left\{[\{1,4\}]^{\uparrow}, [\{1,2,4\}]^{\uparrow}, [\{2,4\}]^{\uparrow}\right\}, \\ \left\{[\{1,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow}, [\{1,3\}]^{\uparrow}\right\} \end{pmatrix}$$

4. Discussion. The commuting property can fail in two ways for the exponential. One, there is no $(x, y, z)_{\text{exp}}$ triple in the overlapping exponential subalgebras that meets the inclusion criteria that $a \leq x,\ b \leq y,\ c \leq z$. Two, the $(x, y, z)_{\text{exp}}$ triple in the overlapping exponential subalgebras does not meet the non-intersection criteria that $x \wedge y \wedge z = 0$.

When we verify these results manually, we see that there is only one $(x, y, z)_{\text{exp}}$ triple that meets the inclusion criteria for all four cases of the $(a, b, c)_{\text{exp}}$ triples when we look at them individually,

$$(x,y,z)_{\text{exp}} = \begin{pmatrix} \left\{[\{1,4\}]^{\uparrow}, [\{1,2,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow}, \{1,2,3,4\}\right\}, \\ \left\{\begin{matrix} [\{2,4\}]^{\uparrow}, [\{1,4\}]^{\uparrow}, \{1,2,3,4\}, \\ [\{1,2,4\}]^{\uparrow}, [\{2,3,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow} \end{matrix}\right\}, \\ \left\{\begin{matrix} [\{1,4\}]^{\uparrow}, \{1,2,3,4\}, [\{1,3\}]^{\uparrow}, \\ [\{1,2,4\}]^{\uparrow}, [\{1,2,3\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow} \end{matrix}\right\} \end{pmatrix}.$$

We can easily verify that $(a, b, c)_{1,\text{exp}} \leq (x, y, z)_{\text{exp}}$, $(a, b, c)_{2,\text{exp}} \leq (x, y, z)_{\text{exp}}$, $(a, b, c)_{3,\text{exp}} \leq (x, y, z)_{\text{exp}}$, and $(a, b, c)_{4,\text{exp}} \leq (x, y, z)_{\text{exp}}$.

$$\begin{pmatrix} [\{1,4\}]^{\uparrow}, \\ \left\{[\{1,4\}]^{\uparrow}, [\{1,2,4\}]^{\uparrow}, [\{2,4\}]^{\uparrow}\right\}, \\ \left\{[\{1,2,4\}]^{\uparrow}, [\{1,2,3\}]^{\uparrow}, \{1,2,3,4\}\right\} \end{pmatrix}_{(a,b,c)_{1,\text{exp}}} \leq \begin{pmatrix} \left\{[\{1,4\}]^{\uparrow}, [\{1,2,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow}, \{1,2,3,4\}\right\}, \\ \left\{\begin{matrix} [\{2,4\}]^{\uparrow}, [\{1,4\}]^{\uparrow}, \{1,2,3,4\}, \\ [\{1,2,4\}]^{\uparrow}, [\{2,3,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow} \end{matrix}\right\}, \\ \left\{\begin{matrix} [\{1,4\}]^{\uparrow}, \{1,2,3,4\}, [\{1,3\}]^{\uparrow}, \\ [\{1,2,4\}]^{\uparrow}, [\{1,2,3\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow} \end{matrix}\right\} \end{pmatrix}_{(x,y,z)_{\text{exp}}}$$

$$\begin{pmatrix} [\{1,4\}]^{\uparrow}, \\ \left\{[\{2,3,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow}, \{1,2,3,4\}\right\}, \\ \left\{[\{1,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow}, [\{1,3\}]^{\uparrow}\right\} \end{pmatrix}_{(a,b,c)_{2,\text{exp}}} \leq \begin{pmatrix} \left\{[\{1,4\}]^{\uparrow}, [\{1,2,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow}, \{1,2,3,4\}\right\}, \\ \left\{\begin{matrix} [\{2,4\}]^{\uparrow}, [\{1,4\}]^{\uparrow}, \{1,2,3,4\}, \\ [\{1,2,4\}]^{\uparrow}, [\{2,3,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow} \end{matrix}\right\}, \\ \left\{\begin{matrix} [\{1,4\}]^{\uparrow}, \{1,2,3,4\}, [\{1,3\}]^{\uparrow}, \\ [\{1,2,4\}]^{\uparrow}, [\{1,2,3\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow} \end{matrix}\right\} \end{pmatrix}_{(x,y,z)_{\text{exp}}}$$

$$\begin{pmatrix} [\{1,4\}]^{\uparrow}, \\ \left\{[\{2,3,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow}, \{1,2,3,4\}\right\}, \\ \left\{[\{1,2,4\}]^{\uparrow}, [\{1,2,3\}]^{\uparrow}, \{1,2,3,4\}\right\} \end{pmatrix}_{(a,b,c)_{3,\mathrm{exp}}} \leq \begin{pmatrix} \left\{[\{1,4\}]^{\uparrow}, [\{1,2,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow}, \{1,2,3,4\}\right\}, \\ \left\{ \begin{matrix} [\{2,4\}]^{\uparrow}, [\{1,4\}]^{\uparrow}, \{1,2,3,4\}, \\ [\{1,2,4\}]^{\uparrow}, [\{2,3,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow} \end{matrix} \right\}, \\ \left\{ \begin{matrix} [\{1,4\}]^{\uparrow}, \{1,2,3,4\}, [\{1,3\}]^{\uparrow}, \\ [\{1,2,4\}]^{\uparrow}, [\{1,2,3\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow} \end{matrix} \right\} \end{pmatrix}_{(x,y,z)_{\mathrm{exp}}}$$

$$\begin{pmatrix} \left\{[\{1,2,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow}, \{1,2,3,4\}\right\}, \\ \left\{[\{1,4\}]^{\uparrow}, [\{1,2,4\}]^{\uparrow}, [\{2,4\}]^{\uparrow}\right\}, \\ \left\{[\{1,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow}, [\{1,3\}]^{\uparrow}\right\} \end{pmatrix}_{(a,b,c)_{4,\mathrm{exp}}} \leq \begin{pmatrix} \left\{[\{1,4\}]^{\uparrow}, [\{1,2,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow}, \{1,2,3,4\}\right\}, \\ \left\{ \begin{matrix} [\{2,4\}]^{\uparrow}, [\{1,4\}]^{\uparrow}, \{1,2,3,4\}, \\ [\{1,2,4\}]^{\uparrow}, [\{2,3,4\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow} \end{matrix} \right\}, \\ \left\{ \begin{matrix} [\{1,4\}]^{\uparrow}, \{1,2,3,4\}, [\{1,3\}]^{\uparrow}, \\ [\{1,2,4\}]^{\uparrow}, [\{1,2,3\}]^{\uparrow}, [\{1,3,4\}]^{\uparrow} \end{matrix} \right\} \end{pmatrix}_{(x,y,z)_{\mathrm{exp}}}$$

However, the triple $(x, y, z)_{\mathrm{exp}}$ does not meet the three-way commuting requirement that $x \wedge y \wedge z = 0$. Therefore, the commuting property of the subalgebra triple was destroyed by the exponential functor.

Out of the 286 subalgebra triples that can be generated from the four atom Boolean Algebra, we found many more subalgebra triples that witness Theorem 1.5. We presented one such example in this work and leave it to future researchers to investigate the remaining examples to investigate this phenomenon deeper.

5. Conclusions. We show that the three atom Boolean Algebra does not show the commuting property for its three, non-trivial subalgebras. We provided one example of a subalgebra triple of the four atom Concrete Boolean Algebra that shows that the three-way commuting property, $A_D \trianglelefteq B_D \trianglelefteq C_D$, was destroyed by the exponential functor, $\exp_D(A) \ntrianglelefteq \exp_D(B) \ntrianglelefteq \exp_D(C)$. Therefore, the four atomic Boolean Algebra with the three subalgebras generated in this work, $A, B, C \leq D$, witnesses Theorem 1.5 (*Destroying the Weak Commuting Property in Subalgebra Triples*). By Theorem 1.6 (*Unlimited Inheritance of Weak Commuting Subalgebra Triples*), we know that every other Boolean Algebra having the same or a higher atomic number than four will also witness Theorem 1.5. Therefore, we can confidently conclude that the smallest Boolean Algebra that witnesses Theorem 1.5—where the exponential functor destroys the weak commuting property of subalgebra triples—is the four atom Boolean Algebra.

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Appendix A – Investigation Results for the Three Atom Boolean Algebra

The only non-trivial three atom Boolean Algebra is shown below.

$$D = \left\{ \begin{array}{c} \{1,2,3\}, \\ \{1,2\},\{1,3\},\{2,3\}, \\ \{1\},\{2\},\{3\}, \\ \{\ \ \} \end{array} \right\}$$

Three subalgebras of D, $A, B, C \leq D$, were generated as shown below.

$$A = \langle\{1\}\rangle = \left\{ \begin{array}{c} \{1,2,3\}, \\ \{1\},\{2,3\}, \\ \{\ \ \} \end{array} \right\}$$

$$B = \langle\{2\}\rangle = \left\{ \begin{array}{c} \{1,2,3\}, \\ \{2\},\{1,3\}, \\ \{\ \ \} \end{array} \right\}$$

$$C = \langle\{3\}\rangle = \left\{ \begin{array}{c} \{1,2,3\}, \\ \{3\},\{1,2\}, \\ \{\ \ \} \end{array} \right\}$$

Three overlapping subalgebras, $P, Q, R \leq D$, satisfying Definition 1.18 were generated as follows.

$$P = \langle A \cap (B \cup C)\rangle = \langle\{1,2,3\},\{\ \}\rangle$$

$$= \left\{ \begin{array}{c} \{1,2,3\}, \\ \{\ \ \} \end{array} \right\}$$

$$Q = \langle B \cap (A \cup C)\rangle = \langle\{1,2,3\},\{\ \}\rangle$$

$$= \left\{ \begin{array}{c} \{1,2,3\}, \\ \{\ \ \} \end{array} \right\}$$

$$R = \langle C \cap (A \cup B) \rangle = \langle \{1,2,3\}, \{\ \} \rangle$$

$$= \left\{ \begin{matrix} \{1,2,3\}, \\ \{\ \ \} \end{matrix} \right\}$$

We observe that every triple $(a, b, c) \in A$ x B x C such that $a \wedge b \wedge c = 0$ finds an element in the triple $(x, y, z) \in P$ x Q x R such that $a \leq x, b \leq y, c \leq z$. However, the condition that $x \wedge y \wedge z = 0$ is not met, so the three atom Boolean Algebra with its only three, non-trivial subalgebras, does not three-way commute, $A_D \ntrianglelefteq B_D \ntrianglelefteq C_D$. Since it does not three-way commute, it is not necessary to apply the exponential functor and we directly conclude that the three atom Boolean Algebra does not witness Theorem 1.5 (*Destroying the Weak Commuting Property in Subalgebra Triples*).

Appendix B Three-Way Commuting Triples

We verified the 3-way commuting property, $A_D \trianglelefteq B_D \trianglelefteq C_D$, and below we list every triple $(a, b, c) \in A$ x B x C such that $a \wedge b \wedge c = 0$.

({4}, {1, 2}, {3, 4}),	({4}, {1, 2}, {2}),	({4}, {1, 2}, {1}),	({4}, {4}, {2}),
({4}, {4}, {1}),	({4}, {3}, {3, 4}),	({1}, {3}, {2}),	({4}, {3}, {1}),
({4}, {3}, {2}),	({2, 3}, {1, 2}, {1}),	({2, 3}, {4}, {3, 4}),	({2, 3}, {4}, {2}),
({2, 3}, {4}, {1}),	({2, 3}, {3}, {2}),	({2, 3}, {3}, {1}),	({1}, {3}, {1}),
({1}, {1, 2}, {2}),	({1}, {4}, {3, 4}),	({1}, {4}, {2}),	({1}, {4}, {1}),
({1}, {3}, {3, 4}),	({2, 3}, {1, 2}, {3, 4}),		({1}, {1, 2}, {3, 4})

Below we list every triple $(x, y, z) \in P$ x Q x R such that $x \wedge y \wedge z = 0$.

({4}, {1, 2}, {3, 4}),	({4}, {1, 2}, {2}),	({4}, {1, 2}, {1}),	({4}, {4}, {2}),
({4}, {4}, {1}),	({4}, {3}, {3, 4}),	({1}, {3}, {2}),	({4}, {3}, {1}),
({4}, {3}, {2}),	({2, 3}, {1, 2}, {1}),	({2, 3}, {4}, {3, 4}),	({2, 3}, {4}, {2}),
({2, 3}, {4}, {1}),	({2, 3}, {3}, {2}),	({2, 3}, {3}, {1}),	({1}, {3}, {1})
({1}, {1, 2}, {2}),	({1}, {4}, {3, 4}),	({1}, {4}, {2}),	({1}, {4}, {1}),
({1}, {3}, {3, 4}),	({2, 3}, {1, 2}, {3, 4}),		({1}, {1, 2}, {3, 4})

It is easily verifiable that for every triplet (a, b, c) there exists a triplet (x, y, z) such that $a \leq x$, $b \leq y$, and $c \leq z$.

Appendix C – Complete List of Atoms of the Exponential Subalgebras and Their Overlaps

$$
\exp_{\mathrm{D}}(A)_{\mathrm{atoms}} = \left\{ \begin{array}{c}
\big\{\{2,4\},\{1,2,3,4\},\{1,4\},\{2,3,4\},\{1,2,4\},\{3,4\},\{4\},\{1,3,4\}\big\}, \\
\left\{ \begin{array}{c} \big\{\{1,2,3,4\},\{1,2,3\}\big\}, \\ \big\{\{1,2,3,4\},\{1,2\},\{1,2,3\},\{1,2,4\}\big\}, \\ \big\{\{1,2,3,4\},\{1,3,4\},\{1,3\},\{1,2,3\}\big\} \end{array} \right\}, \\
\big\{\{1,2,3,4\},\{1,3,4\},\{1,4\},\{1,2,4\}\big\}, \\
\left\{ \begin{array}{c} \big\{\{1,2,3,4\},\{1,2,4\}\big\}, \\ \big\{\{1,2,3,4\},\{1,3,4\}\big\}, \\ \{1,2,3,4\} \end{array} \right\}, \\
\left\{ \begin{array}{c} \big\{\{2,4\},\{1,2,3,4\},\{1,2\},\{2,3,4\},\{1,2,4\},\{2,3\},\{1,2,3\},\{2\}\big\}, \\ \big\{\{3\},\{1,2,3,4\},\{2,3,4\},\{3,4\},\{1,3,4\},\{2,3\},\{1,3\},\{1,2,3\}\big\}, \\ \big\{\{1,2,3,4\},\{2,3\},\{1,2,3\},\{2,3,4\}\big\} \end{array} \right\}, \\
\left\{ \begin{array}{c} \big\{\{1,2,3,4\},\{2,3,4\},\{2,4\},\{1,2,4\}\big\}, \\ \big\{\{1,2,3,4\},\{2,3,4\}\big\}, \\ \big\{\{1,2,3,4\},\{3,4\},\{1,3,4\},\{2,3,4\}\big\} \end{array} \right\}, \\
\big\{\{1,2,3,4\},\{1,2\},\{1,4\},\{1,2,4\},\{1,3,4\},\{1\},\{1,3\},\{1,2,3\}\big\}
\end{array} \right\}
$$

$$
\exp_D(B)_{\text{atoms}} = \left\{ \begin{array}{c}
\{\{1,2,3,4\},\{3,4\},\{1,3,4\},\{2,3,4\}\}, \\[1em]
\left\{ \begin{array}{c} \{\{1,2,3,4\},\{1,3,4\},\{1,4\},\{1,2,4\}\}, \\ \{\{1,2,3,4\},\{1,2,4\}\}, \\ \{\{1,2,3,4\},\{2,3,4\},\{2,4\},\{1,2,4\}\} \end{array} \right\}, \\[1em]
\{\{2,4\},\{1,2,3,4\},\{1,4\},\{2,3,4\},\{1,2,4\},\{3,4\},\{4\},\{1,3,4\}\}, \\[1em]
\left\{ \begin{array}{c} \{\{1,2,3,4\},\{1,3,4\},\{1,3\},\{1,2,3\}\}, \\ \{\{1,2,3,4\},\{1,2,3\}\}, \\ \{\{1,2,3,4\},\{2,3\},\{1,2,3\},\{2,3,4\}\} \end{array} \right\}, \\
\left\{ \begin{array}{c} \{\{1,2,3,4\},\{1,2\},\{1,4\},\{1,2,4\},\{1,3,4\},\{1\},\{1,3\},\{1,2,3\}\}, \\ \{\{2,4\},\{1,2,3,4\},\{1,2\},\{2,3,4\},\{1,2,4\},\{2,3\},\{1,2,3\},\{2\}\}, \\ \{\{1,2,3,4\},\{1,2\},\{1,2,3\},\{1,2,4\}\} \end{array} \right\}, \\[1em]
\left\{ \begin{array}{c} \{\{1,2,3,4\},\{2,3,4\}\}, \\ \{\{1,2,3,4\},\{1,3,4\}\}, \\ \{1,2,3,4\} \end{array} \right\}, \\[1em]
\{\{3\},\{1,2,3,4\},\{2,3,4\},\{3,4\},\{1,3,4\},\{2,3\},\{1,3\},\{1,2,3\}\}
\end{array} \right\}
$$

$$\exp_D(C)_{\mathrm{atoms}} = \left\{\begin{array}{c} \left\{\begin{array}{c}\{1,2,3,4\},\\ \{1,2\},\{1,2,3\},\\ \{1,2,4\}\end{array}\right\}, \\[1em] \left\{\begin{array}{c}\{\{1,2,3,4\},\{1,3,4\},\{1,4\},\{1,2,4\}\},\\ \{\{1,2,3,4\},\{1,3,4\}\},\\ \{\{1,2,3,4\},\{1,3,4\},\{1,3\},\{1,2,3\}\}\end{array}\right\}, \\[1em] \left\{\begin{array}{c}\{\{2,4\},\{1,2,3,4\},\{1,4\},\{2,3,4\},\{1,2,4\},\{3,4\},\{4\},\{1,3,4\}\},\\ \{\{1,2,3,4\},\{3,4\},\{1,3,4\},\{2,3,4\}\},\\ \{\{3\},\{1,2,3,4\},\{2,3,4\},\{3,4\},\{1,3,4\},\{2,3\},\{1,3\},\{1,2,3\}\}\end{array}\right\}, \\[1em] \left\{\begin{array}{c}\{\{1,2,3,4\},\{1,2,4\}\},\\ \{\{1,2,3,4\},\{1,2,3\}\},\\ \{1,2,3,4\}\end{array}\right\}, \\[1em] \left\{\begin{array}{c}\{\{1,2,3,4\},\{2,3,4\}\},\\ \{\{1,2,3,4\},\{2,3,4\},\{2,4\},\{1,2,4\}\},\\ \{\{1,2,3,4\},\{2,3\},\{1,2,3\},\{2,3,4\}\}\end{array}\right\} \\[1em] \left\{\begin{array}{c}\{1,2,3,4\},\{1,2\},\{1,4\},\{1,2,4\},\\ \{1,3,4\},\{1\},\{1,3\},\{1,2,3\}\end{array}\right\}, \\[1em] \left\{\begin{array}{c}\{2,4\},\{1,2,3,4\},\{1,2\},\{2,3,4\},\\ \{1,2,4\},\{2,3\},\{1,2,3\},\{2\}\end{array}\right\} \end{array}\right\}$$

The three overlapping exponential subalgebras satisfying Definition 1.18 were generated as follows.

$$\langle \exp_D(A) \cap [\exp_D(B) \cup \exp_D(C)] \rangle_{\text{atoms}}$$

$$= \left\{ \begin{array}{c} \left\{ \begin{array}{c} \{\{1,2,3,4\},\{1,3,4\},\{1,4\},\{1,2,4\}\}, \\ \{\{1,2,3,4\},\{1,2,4\}\}, \\ \{\{1,2,3,4\},\{1,3,4\}\}, \\ \{1,2,3,4\} \end{array} \right\}, \\ \{\{2,4\},\{1,2,3,4\},\{1,4\},\{2,3,4\},\{1,2,4\},\{3,4\},\{4\},\{1,3,4\}\}, \\ \left\{ \begin{array}{c} \{\{1,2,3,4\},\{1,2,3\}\}, \\ \{\{1,2,3,4\},\{1,2\},\{1,2,3\},\{1,2,4\}\}, \\ \{\{1,2,3,4\},\{1,3,4\},\{1,3\},\{1,2,3\}\} \end{array} \right\}, \\ \left\{ \begin{array}{c} \{\{2,4\},\{1,2,3,4\},\{1,2\},\{2,3,4\},\{1,2,4\},\{2,3\},\{1,2,3\},\{2\}\}, \\ \{\{3\},\{1,2,3,4\},\{2,3,4\},\{3,4\},\{1,3,4\},\{2,3\},\{1,3\},\{1,2,3\}\}, \\ \{\{1,2,3,4\},\{2,3\},\{1,2,3\},\{2,3,4\}\} \end{array} \right\}, \\ \left\{ \begin{array}{c} \{\{1,2,3,4\},\{2,3,4\}\}, \\ \{\{1,2,3,4\},\{3,4\},\{1,3,4\},\{2,3,4\}\}, \\ \{\{1,2,3,4\},\{2,3,4\},\{2,4\},\{1,2,4\}\} \end{array} \right\}, \\ \{\{1,2,3,4\},\{1,2\},\{1,4\},\{1,2,4\},\{1,3,4\},\{1\},\{1,3\},\{1,2,3\}\}, \end{array} \right\}$$

$$\langle \exp_D(B) \cap [\exp_D(A) \cup \exp_D(C)] \rangle_{\text{atoms}}$$

$$= \left\{ \begin{array}{c} \{\{1,2,3,4\},\{3,4\},\{1,3,4\},\{2,3,4\}\}, \\ \{\{2,4\},\{1,2,3,4\},\{1,4\},\{2,3,4\},\{1,2,4\},\{3,4\},\{4\},\{1,3,4\}\}, \\ \left\{ \begin{array}{c} \{\{1,2,3,4\},\{1,2,3\}\}, \\ \{\{1,2,3,4\},\{1,3,4\},\{1,3\},\{1,2,3\}\}, \\ \{\{1,2,3,4\},\{2,3\},\{1,2,3\},\{2,3,4\}\} \end{array} \right\}, \\ \left\{ \begin{array}{c} \{1,2,3,4\}, \\ \{\{1,2,3,4\},\{2,3,4\},\{2,4\},\{1,2,4\}\}, \\ \{\{1,2,3,4\},\{1,3,4\},\{1,4\},\{1,2,4\}\}, \\ \{\{1,2,3,4\},\{1,2,4\}\}, \\ \{\{1,2,3,4\},\{2,3,4\}\}, \\ \{\{1,2,3,4\},\{1,3,4\}\} \end{array} \right\}, \\ \left\{ \begin{array}{c} \{\{1,2,3,4\},\{1,2\},\{1,4\},\{1,2,4\},\{1,3,4\},\{1\},\{1,3\},\{1,2,3\}\}, \\ \{\{2,4\},\{1,2,3,4\},\{1,2\},\{2,3,4\},\{1,2,4\},\{2,3\},\{1,2,3\},\{2\}\}, \\ \{\{1,2,3,4\},\{1,2\},\{1,2,3\},\{1,2,4\}\} \end{array} \right\}, \\ \{\{3\},\{1,2,3,4\},\{2,3,4\},\{3,4\},\{1,3,4\},\{2,3\},\{1,3\},\{1,2,3\}\} \end{array} \right\}$$

$$\langle \exp_D(C) \cap [\exp_D(A) \cup \exp_D(B)] \rangle_{\text{atoms}}$$

$$= \left\{ \begin{array}{c} \{\{1,2,3,4\},\{1,2\},\{1,2,3\},\{1,2,4\}\}, \\ \\ \left\{ \begin{array}{c} \{\{2,4\},\{1,2,3,4\},\{1,4\},\{2,3,4\},\{1,2,4\},\{3,4\},\{4\},\{1,3,4\}\}, \\ \{\{1,2,3,4\},\{3,4\},\{1,3,4\},\{2,3,4\}\}, \\ \{\{3\},\{1,2,3,4\},\{2,3,4\},\{3,4\},\{1,3,4\},\{2,3\},\{1,3\},\{1,2,3\}\} \end{array} \right\}, \\ \left\{ \begin{array}{c} \{\{1,2,3,4\},\{2,3,4\}\}, \\ \{\{1,2,3,4\},\{2,3,4\},\{2,4\},\{1,2,4\}\}, \\ \{\{1,2,3,4\},\{2,3\},\{1,2,3\},\{2,3,4\}\} \end{array} \right\}, \\ \{\{1,2,3,4\},\{1,2\},\{1,4\},\{1,2,4\},\{1,3,4\},\{1\},\{1,3\},\{1,2,3\}\}, \\ \\ \left\{ \begin{array}{c} \{1,2,3,4\}, \\ \{\{1,2,3,4\},\{1,3,4\},\{1,4\},\{1,2,4\}\}, \\ \{\{1,2,3,4\},\{1,3,4\},\{1,3\},\{1,2,3\}\}, \\ \{\{1,2,3,4\},\{1,2,4\}\}, \\ \{\{1,2,3,4\},\{1,2,3\}\}, \\ \{\{1,2,3,4\},\{1,3,4\}\} \end{array} \right\}, \\ \\ \{\{2,4\},\{1,2,3,4\},\{1,2\},\{2,3,4\},\{1,2,4\},\{2,3\},\{1,2,3\},\{2\}\} \end{array} \right\}$$

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