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The Ramification Group Filtrations of Certain Function Field Extensions

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THE RAMIFICATION GROUP FILTRATIONS OF CERTAIN FUNCTION FIELD EXTENSIONS

A Thesis

by

JEFFREY ANTHONY CASTAÑEDA

Submitted to Texas A&M International University
in partial fulfillment of the requirements
for the degree of

MASTER OF SCIENCE

December 2014

Major Subject: Mathematics

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ABSTRACT

The Ramification Group Filtrations of Certain Function Field Extensions (December 2014)

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Chair of Committee: Dr. Qingquan Wu

We investigate the ramification group filtration of a Galois extension of function fields, if the Galois group satisfies a certain intersection property. For finite groups, this property is implied by having only elementary abelian Sylow p -subgroups. Note that such groups could be non-abelian. We show how the problem can be reduced to the totally wild ramified case on a p -extension. Our methodology is based on an intimate relationship between the ramification groups of the field extension and those of all degree p sub-extensions. Not only do we confirm that the Hasse-Arf property holds in this setting, but we also prove that the Hasse-Arf divisibility result is the best possible by explicit calculations of the quotients, which are expressed in terms of the different exponents of all those degree p sub-extensions.

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TABLE OF CONTENTS

	Page
ABSTRACT	iv
ACKNOWLEDGMENTS	v
TABLE OF CONTENTS	vi
CHAPTER	
I INTRODUCTION	1
II NOTATION AND TERMINOLOGY	3
III REDUCTION TO THE TOTALLY WILDLY RAMIFIED CASE	6
IV MAIN RESULTS	8
V HASSE-ARF PROPERTY	12
VI CONCLUSION	14
REFERENCES	15
VITA	17

CHAPTER I

INTRODUCTION

When investigating algebraic number fields and function fields, Hilbert ramification theory is a convenient tool, especially in the study of wild ramifications. Fix a function field K over a perfect constant field k with a place \mathcal{P} , and let L be a Galois extension of K with a place \mathfrak{P} lying over \mathcal{P} . We investigate how the ramification group filtration of $\mathfrak{P}|\mathcal{P}$ is related to the ramification group filtration of $\mathfrak{P}_m|\mathcal{P}$, where \mathfrak{P}_m is a place of some intermediate field $K \subseteq M \subseteq L$, so that \mathfrak{P} lies over \mathfrak{P}_m and $[M : K] = p$ for some prime number p .

We first analyze how and why we can simplify the problem to the setting when $\mathfrak{P}|\mathcal{P}$ is totally wildly ramified, i.e. $[L : K] = p^n$, where $p > 0$ is the characteristic of k , n is some positive integer, and the ramification index $e(\mathfrak{P}|\mathcal{P}) = p^n$.

Next we study how the ramification group filtration of $\mathfrak{P}|\mathcal{P}$ is closely related to the ramification group filtration of $\mathfrak{P}_m|\mathcal{P}$ for various degree p to all those intermediate fields M such that $[M : K] = p$. This relation is close if an intersection property (5) is assumed about $\text{Gal}(L/K)$, which is satisfied by many abelian and non-abelian groups.

To prove such a relationship, we first prove a preliminary result, which states that the number of jumps in the ramification group filtration is equal to the number of pairwise distinct different exponents of the corresponding place extensions over all possible degree p intermediate extensions M/K . This equality is significant since the degree p intermediate extensions are considerably easier to investigate than the whole extension L/K . Nonetheless, we will show that these different exponents are closely related to those quotients given in the Hasse-Arf property, which we will show to be true.

Furthermore, we study the relationship by applying the equality given by the transitivity of differentials, where the different exponents are computed via the Hilbert Different Formulae applied to various field extension settings. These equalities lead to linear equations on the indices where the jumps of the ramification group filtration on L/K occur. With the intersection property assumption, we show that the number of such linear equations is equal to the number of such indices as the variables of these equations. Hence, we can expect a unique solution. In fact, we can solve these linear equations explicitly to give closed-form formulae for the indices since the coefficient matrix of the linear equations is triangular.

The journal model used is from the *International Journal of Number Theory*.

The academic literature on ramification group filtration is extensive. A good introduction is [18], where Herbrand's upper numbering is introduced. See [8] for an introduction without the use of cohomologies. The ramification groups are studied in [17] using class field theory. For an approach using Herbrand functions and without using class field theory, see [23]. In [13], Maus showed certain properties of a group filtration are sufficient to guarantee it to be the ramification group filtration of a certain extension of complete discrete valuation fields. In [15], the asymptotic behavior of quotients given by the Hasse-Arf property is studied. The paper [14] is a collection of many results from Maus's Ph.D. thesis without proofs.

The ramification group filtration is known to satisfy the Hasse-Arf property [10, 11, 2] if the Galois group is abelian. However, the property may fail if the Galois group is not abelian. One such example is the Galois closure of a cyclotomic field over the rationals [21]. In [6], it is shown that the ramification group filtration of a wildly ramified prime \mathfrak{p} is uniquely determined by the \mathfrak{p} -adic valuation of the discriminant of the field extension L/K , when both the field extension degree and the residue characteristic of \mathfrak{p} are equal to a prime number. When the Galois group is elementary abelian, the Galois module structure of certain ideals is related to the ramification group filtration, see [3, 4, 5]. Such a relation is investigated when the Galois group is quaternion [7], hence non-abelian.

For the function field extension setting, the wildly ramified case was studied in Artin-Schreier-Witt extensions, see [20]. The elementary abelian extension of Galois group $\mathbb{Z}_p \times \mathbb{Z}_p$ is investigated in [1] and [22]. It should be mentioned that the idea of utilizing transitivity of differentials and Hilbert's Different Formula to investigate the ramification groups is used in [9], where the Hasse-Arf property for elementary abelian extensions of function fields is proved. In Roberts' review (MR2436331) of [9], it is shown that the proof can be very short if "some upper numbering system and its basic formalism" is applied. We take a similar approach to [9], but we further explore the arithmetic and linear algebra provided by the application of transitivity of differentials and Hilbert's Different Formula. Our objective in this thesis paper is to generalize these results to function field extensions with Galois groups satisfying a certain intersection property which is true for elementary abelian groups.

CHAPTER II

NOTATION AND TERMINOLOGY

A good introduction for the notation can be found in [16] or [19]. The following notations we used throughout this thesis are

- k is a perfect field of characteristic $p > 0$;
- K is a function field with constant field k ;
- \mathcal{P} is a place of K ;
- $v_{\mathcal{P}} : K \mapsto \mathbb{Z} \cup \{\infty\}$ is the (surjective) discrete valuation corresponding to \mathcal{P} ;
- $\mathcal{O}_{\mathcal{P}} = \{\alpha \in K \mid v_{\mathcal{P}}(\alpha) \geq 0\}$ is the valuation ring corresponding to \mathcal{P} .

Next, we introduce some terminology, which can be found in [19]. Let L be a field and K a subfield of L such that $K \subseteq L$, then L/K is a field extension. If there is a polynomial $f(X) \neq 0$ such that $f(X) \in K[X]$ where $f(\alpha) = 0$, then an element $\alpha \in L$ is algebraic over K . If all elements $\alpha \in L$ are algebraic over K , then the extension L/K is algebraic. Let $f(X) \in K[X]$ be a monic polynomial of degree greater than or equal to 1. Then $f(X)$ splits into linear factors over K as $\prod_{i=1}^d (X - x_i)$. If $x_i \neq x_j$ for all $i \neq j$, we called $f(X)$ separable. Moreover, an element $x \in L$ is called separable over K if its minimal polynomial $f(X) \in K[X]$ is a separable polynomial. If all $x \in L$ are separable over K , then L/K is a separable extension. If all algebraic extensions of K are separable, then K is perfect.

Let $e(\mathfrak{P}|\mathcal{P})$ and $d(\mathfrak{P}|\mathcal{P})$ be the ramification index and different exponent of $\mathfrak{P}|\mathcal{P}$, respectively. The ramification index $e(\mathfrak{P}|\mathcal{P})$ is an integer such that $v_{\mathfrak{P}}(\alpha) = e(\mathfrak{P}|\mathcal{P})v_{\mathcal{P}}(\alpha)$ for some $\alpha \in L$. Furthermore, if the $e(\mathfrak{P}|\mathcal{P}) > 1$, then $\mathfrak{P}|\mathcal{P}$ is ramified; otherwise it is unramified.

Let L/K be an algebraic extension of function fields of a perfect field of constant k , \mathcal{P} a place of L , and \mathfrak{P} a place of K . We write $\mathfrak{P}|\mathcal{P}$ as a place \mathfrak{P} of L lying above \mathcal{P} a place of K . Then, $\mathfrak{P}|\mathcal{P}$ is tamely ramified if $\text{char}(K) \nmid e(\mathfrak{P}|\mathcal{P})$ for some $\mathfrak{P}|\mathcal{P}$. Otherwise, $\mathfrak{P}|\mathcal{P}$ is wildly ramified. If $e(\mathfrak{P}|\mathcal{P}) = [L : K]$ and there is only one extension \mathfrak{P} of \mathcal{P} , then \mathcal{P} totally ramifies in L/K . Next, we let $\text{Aut}(L/K)$ denote as the group of automorphisms of L over K such that for $\sigma \in \text{Aut}(L/K)$ is a K -isomorphism of L into L . If the order of $\text{Aut}(L/K)$ is $[L : K]$, then L/K is called a Galois extension. We also write $\text{Gal}(L/K) = \text{Aut}(L/K)$ as the Galois group of L over K . If L/K is a Galois extension, the ramification groups of $\mathfrak{P}|\mathcal{P}$ are given by

$$(1) \quad G_i = G_i(\mathfrak{P}|\mathcal{P}) = \{\sigma \in \text{Gal}(L/K) \mid v_{\mathfrak{P}}(t^\sigma - t) \geq i + 1 \text{ for all } t \in \mathcal{O}_{\mathfrak{P}}\}$$

for $i \geq 0$. The connection between these groups and the different exponent is shown in Hilbert's different formula (see for example Theorem 3.8.7, p. 136, of [19]). For the purpose of completeness, we state Theorem 3.8.7 as the following:

THEOREM 2.1. *For a Galois extension L/K of algebraic function fields, a place \mathcal{P} in K , and a place \mathfrak{P} in L lying above \mathcal{P} . Then the different exponent $d(\mathfrak{P}|\mathcal{P})$ is*

$$(2) \quad d(\mathfrak{P}|\mathcal{P}) = \sum_{i=0}^{\infty} (\#G_i(\mathfrak{P}|\mathcal{P}) - 1).$$

We also recall the transitivity of the ramification index and the different exponent. If $K \subseteq F \subseteq L$ are function fields, \mathfrak{P} a place of L , $\mathfrak{P}_{\mathfrak{F}} = \mathfrak{P} \cap F$, and $\mathcal{P} = \mathfrak{P}_{\mathfrak{F}} \cap K$, then

$$(3) \quad e(\mathfrak{P}|\mathcal{P}) = e(\mathfrak{P}|\mathfrak{P}_{\mathfrak{F}})e(\mathfrak{P}_{\mathfrak{F}}|\mathcal{P}),$$

and we have transitivity of differentials:

$$(4) \quad d(\mathfrak{P}|\mathcal{P}) = e(\mathfrak{P}|\mathfrak{P}_{\mathfrak{F}})d(\mathfrak{P}_{\mathfrak{F}}|\mathcal{P}) + d(\mathfrak{P}|\mathfrak{P}_{\mathfrak{F}}).$$

Henceforth, we assume that all non-trivial Sylow p -subgroups H_p of the Galois group $\text{Gal}(L/K)$ satisfy the following intersection property.

Assume that $\#H_p = p^n > 1$. Then, for all proper subgroups $F \subsetneq H_p$, the intersection of all order p^{n-1} subgroups of H_p containing F is simply F . That is to say,

$$(5) \quad \bigcap_{H \supseteq F, \#H=p^{n-1}} H = F.$$

It is easy to verify that all elementary abelian p -groups of order p^n satisfy this intersection property. Furthermore, we will also refer to Theorem 4.7, p. 39, of [12]. Theorem 4.7 states the following:

THEOREM 2.2. *For H and H' be finite subgroups of a group G . Then*

$$\#HH' = \#H\#H' / \#(H \cap H').$$

Lastly, we also refer to Proposition 3.7.8, p. 127, of [19] this is known as the Artin-Schreier Extensions. We state Proposition 3.7.8 as the following:

PROPOSITION 2.3. *For L/K be an algebraic function field of characteristic $p > 0$. Suppose that $u \in L$ is an element which satisfies the following condition:*

$$u \neq \gamma^p - \gamma \text{ for all } \gamma \in L.$$

Let

$$L' = L(\alpha) \text{ with } \alpha^p - \alpha = u.$$

Such an extension L'/L is called an Artin-Schreier extension of L . For \mathcal{P} be a place of L we define the integer $m_{\mathcal{P}}$ by

$$m_{\mathcal{P}} := \begin{cases} m & \text{if there is an element } \beta \in L \text{ satisfying } v_{\mathcal{P}}(u - (\beta^p - \beta)) = -m < 0 \text{ and} \\ & m \not\equiv 0 \pmod{p}, \\ -1 & \text{if } v_{\mathcal{P}}(u - (\beta^p - \beta)) \geq 0 \text{ for some } \beta \in L. \end{cases}$$

Then \mathcal{P} is totally ramified in L'/L if and only if $m_{\mathcal{P}} > 0$, where \mathfrak{P} is the unique place of L' lying over \mathcal{P} .

Then the different exponent of $d(\mathfrak{P}|\mathcal{P})$ is given by

$$d(\mathfrak{P}|\mathcal{P}) = (p-1)(m_{\mathcal{P}} + 1).$$

Finally, we provide an illustration by applying Proposition 2.3 that $d_i \not\equiv 0 \pmod{p}$ and the strictly increasing property of d_i are the only two restrictions of the sequence of positive integers. Consider a uniformizer t of \mathcal{P} and a positive integer m such that $p \nmid m$. Set $u = t^{-m}$ to show that $u \neq \gamma^p - \gamma$ for any $\gamma \in L$. If $u = \gamma^p - \gamma$, then $v_{\mathcal{P}}(u) = -m < 0$. We consider two cases. If $v_{\mathcal{P}}(\gamma) \geq 0$, then $v_{\mathcal{P}}(\gamma^p - \gamma) \geq \min \{v_{\mathcal{P}}(\gamma^p), v_{\mathcal{P}}(\gamma)\} \geq 0$ by the general inequality (see for example definition 1.1.9, p. 4, part 4, of [19]). If $v_{\mathcal{P}}(\gamma) < 0$, then $v_{\mathcal{P}}(\gamma^p - \gamma) = v_{\mathcal{P}}(\gamma^p)$ by the Strict Triangle Inequality (see for example Lemma 1.1.11, p. 5, of [19]). Hence, this implies $p \mid v_{\mathcal{P}}(\gamma^p - \gamma) = pv_{\mathcal{P}}(\gamma)$. However, $p \nmid v_{\mathcal{P}}(u) = -m$, which is a contradiction. Therefore, the function field constructed by $L' = L(\alpha)$ with $\alpha^p - \alpha = u$ is an Artin-Schreier extension of L for $m_{\mathcal{P}}$ defined to be m with any prescribed different exponent $d(\mathfrak{P}|\mathcal{P}) = (p-1)(m_{\mathcal{P}} + 1)$.

CHAPTER III

REDUCTION TO THE TOTALLY WILDLY RAMIFIED CASE

Let L be a Galois extension field of K , \mathfrak{P} a place of L , and $\mathcal{P} = \mathfrak{P} \cap K$. Our goal in this chapter is to reduce the ramification groups $G_i(\mathfrak{P}|\mathcal{P})$ calculation to the case that $\mathfrak{P}|\mathcal{P}$ is totally wildly ramified and the Galois group $\text{Gal}(L/K)$ is a p -group for a certain prime number p .

LEMMA 3.1. *Let L/K be a Galois extension of a function field, \mathfrak{P} a place of L , $\mathcal{P} = \mathfrak{P} \cap K$, and $\mathfrak{P}_m = \mathfrak{P} \cap M$, where M is the inertia field of \mathcal{P} in L/K . Then, $G_i(\mathfrak{P}|\mathcal{P}) = G_i(\mathfrak{P}|\mathfrak{P}_m)$ for every $i \geq 0$.*

PROOF. Applying (4) to the field extension tower $L/M/K$, we have

$$(6) \quad d(\mathfrak{P}|\mathcal{P}) = e(\mathfrak{P}|\mathfrak{P}_m)d(\mathfrak{P}_m|\mathcal{P}) + d(\mathfrak{P}|\mathfrak{P}_m).$$

However, $d(\mathfrak{P}_m|\mathcal{P}) = 0$ since \mathcal{P} is unramified in M/K , so $d(\mathfrak{P}|\mathcal{P}) = d(\mathfrak{P}|\mathfrak{P}_m)$. Now (2) yields

$$(7) \quad d(\mathfrak{P}|\mathcal{P}) = \sum_{i=0}^{\infty} (\#G_i(\mathfrak{P}|\mathcal{P}) - 1)$$

and

$$(8) \quad d(\mathfrak{P}|\mathfrak{P}_m) = \sum_{i=0}^{\infty} (\#G_i(\mathfrak{P}|\mathfrak{P}_m) - 1).$$

By definition (1), it is easy to check that $G_i(\mathfrak{P}|\mathfrak{P}_m)$ is the intersection of $G_i(\mathfrak{P}|\mathcal{P})$ and the Galois group of L/M . In particular, we have $\#G_i(\mathfrak{P}|\mathcal{P}) \geq \#G_i(\mathfrak{P}|\mathfrak{P}_m)$ for all i . Note that $d(\mathfrak{P}|\mathcal{P}) = d(\mathfrak{P}|\mathfrak{P}_m)$, which implies that $\#G_i(\mathfrak{P}|\mathcal{P}) = \#G_i(\mathfrak{P}|\mathfrak{P}_m)$ for all $i \geq 0$ by (7) and (8). Thus, $G_i(\mathfrak{P}|\mathcal{P}) = G_i(\mathfrak{P}|\mathfrak{P}_m)$ for all $i \geq 0$. \square

Next, we want to reduce to the totally wildly ramified case, i.e. $[L : K] = e(\mathfrak{P}|\mathcal{P}) = p^m$, where p is the characteristic of K .

PROPOSITION 3.2. *Let $p > 0$ be the characteristic of the Galois extension of function field L/K , \mathfrak{P} a place of L , $\mathcal{P} = \mathfrak{P} \cap K$, N the inertia field of \mathcal{P} in L/K , $\mathfrak{P}_n = \mathfrak{P} \cap N$, M the intermediate field of L/N corresponding to a Sylow p -subgroup of $\text{Gal}(L/N)$ under Galois correspondence, and $\mathfrak{P}_m = \mathfrak{P} \cap M$. Then \mathfrak{P}_m is totally wildly ramified in the p -extension L/M , and $G_i(\mathfrak{P}|\mathcal{P}) = G_i(\mathfrak{P}|\mathfrak{P}_m)$ for every $i \geq 1$.*

PROOF. By Lemma 3.1, we have $G_i(\mathfrak{P}|\mathcal{P}) = G_i(\mathfrak{P}|\mathfrak{P}_n)$ for every $i \geq 0$. It suffices to show that $G_i(\mathfrak{P}|\mathfrak{P}_n) = G_i(\mathfrak{P}|\mathfrak{P}_m)$ for every $i \geq 1$. Assume that $[L : N] = p^m q$ and $\gcd(p, q) = 1$. We have

$d(\mathfrak{F}_m|\mathfrak{F}_n) = q - 1$ since $\mathfrak{F}_m|\mathfrak{F}_n$ is totally tamely ramified, and we also have $e(\mathfrak{F}|\mathfrak{F}_m) = p^m$. By (6), we have

$$(9) \quad d(\mathfrak{F}|\mathfrak{F}_n) = p^m(q - 1) + d(\mathfrak{F}|\mathfrak{F}_m).$$

Clearly $\#G_0(\mathfrak{F}|\mathfrak{F}_n) = p^m q$ and $\#G_0(\mathfrak{F}|\mathfrak{F}_m) = p^m$. Hence,

$$d(\mathfrak{F}|\mathfrak{F}_n) = p^m q - 1 + \sum_{i=1}^{\infty} (\#G_i(\mathfrak{F}|\mathfrak{F}_n) - 1)$$

and

$$d(\mathfrak{F}|\mathfrak{F}_m) = p^m - 1 + \sum_{i=1}^{\infty} (\#G_i(\mathfrak{F}|\mathfrak{F}_m) - 1)$$

by (7) and (8). Substituting these two equalities into (9), we have

$$\sum_{i=1}^{\infty} (\#G_i(\mathfrak{F}|\mathfrak{F}_n) - 1) = \sum_{i=1}^{\infty} (\#G_i(\mathfrak{F}|\mathfrak{F}_m) - 1),$$

which implies $G_i(\mathfrak{F}|\mathfrak{F}_n) = G_i(\mathfrak{F}|\mathfrak{F}_m)$ for $i \geq 1$ since $G_i(\mathfrak{F}|\mathfrak{F}_n) \geq G_i(\mathfrak{F}|\mathfrak{F}_m)$ for all i . \square

CHAPTER IV

MAIN RESULTS

Henceforth, let L/K be a Galois extension whose Galois group is a p -group, where p is the characteristic of K . Set t to be the number of distinct $d(\mathfrak{P}_m|\mathcal{P})$, where M runs through all degree p intermediate fields M of L/K , and $\mathfrak{P}_m = \mathfrak{P} \cap M$. We assume that the ramification groups of $\mathfrak{P}|\mathcal{P}$ are

$$(10) \quad G_0 = \cdots = G_{m_0} \supsetneq G_{m_0+1} = \cdots = G_{m_1} \supsetneq G_{m_1+1} = \cdots = G_{m_{l-1}} \supsetneq G_{m_{l-1}+1} = \{\text{Id}\}.$$

We let $\#G_{m_i} = p^{n_i}$ for $0 \leq i \leq l-1$. Then $p^{n_0} = p^n = [L : K]$. In order to investigate G_i , we need to know l, n_i , and m_i .

First, we claim that the number of jumps in the ramification groups G_i is the number of distinct different exponents $d(\mathfrak{P}_m|\mathcal{P})$; this is, $l = t$. Note that a jump means an index where a group in the ramification filtration contains the next one properly. Before we prove the claim, we need a lemma.

LEMMA 4.1. *Let G be a p -group of order $p^n > 1$, $H < G$ a subgroup of order p^{n-1} , and $H' < G$ a subgroup such that $H \not\supseteq H'$. Then $\#(H \cap H') = \frac{\#H}{p}$.*

PROOF. By Theorem 2.2, we have $\#(HH') = \frac{\#H\#H'}{\#(H \cap H')}$. In particular, $\#(HH')$ is a power of p . Since $H \not\supseteq H'$, we know that $p^{n-1} < \#(HH') \leq p^n$. Hence, $\#(HH') = p^n$. The result follows by substituting $\#(HH') = p^n$, $\#H = p^{n-1}$ into the equality $\#(HH') = \frac{\#H\#H'}{\#(H \cap H')}$. \square

Furthermore, by (4) we know that $d(\mathfrak{P}|\mathcal{P}) = d(\mathfrak{P}_m|\mathcal{P})p^{n-1} + d(\mathfrak{P}|\mathfrak{P}_m)$, where t distinct $d(\mathfrak{P}_m|\mathcal{P})$ values imply t distinct $d(\mathfrak{P}|\mathfrak{P}_m)$. Let $d(\mathfrak{P}|\mathfrak{P}_m) = \sum_{i=0}^{\infty} (\#G_i(\mathfrak{P}|\mathfrak{P}_m) - 1)$, where $G_i(\mathfrak{P}|\mathfrak{P}_m) = H_m \cap G_i(\mathfrak{P}|\mathcal{P})$ such that $\#H_m = p^{n-1}$.

Now, for every jump $G_{m_i} \supsetneq G_{m_i+1}$ for $0 \leq i \leq l-1$, we want to know if there exists $H_m < G$ such that $\#H_m = p^{n-1}$, $H_m \supseteq G_{m_i+1}$ but $H_m \not\supseteq G_{m_i}$. In other words, we want to know if there exists an order p^{n-1} subgroup H_m which faithfully reveals a jump wherever it occurs in the ramification group filtration. This is not a trivial question since a jump can be hidden if no such H_m can be found. The question is clarified by the following lemma:

LEMMA 4.2. *Let G be a p -group satisfying property (5). Then for any two subgroups $F_1 \supsetneq F_2$ of G , there exists a subgroup H of G such that $\#H = p^{n-1}$, $H \supseteq F_2$ but $H \not\supseteq F_1$.*

PROOF. By way of contradiction, assume that the result is false. Then we can find subgroups $F_1 \supsetneq F_2$, and for all subgroups H of order p^{n-1} , $H \supseteq F_2$ implies $H \supseteq F_1$. Thus, the set $\{H < G \mid \#H = p^{n-1}, H \supseteq F_2\}$

$F_1\} = \{H < G \mid \#H = p^{n-1}, H \supseteq F_2\}$. Hence,

$$F_1 = \bigcap_{H \supseteq F_1, \#H=p^{n-1}} H = \bigcap_{H \supseteq F_2, \#H=p^{n-1}} H = F_2,$$

a contradiction by (5). \square

Now we are ready to prove the result $l = t$.

PROPOSITION 4.3. *Let L/K be a Galois extension of function fields whose Galois group is a p -group satisfying (5), where p is the characteristic of K . Let $\mathfrak{P} \mid \mathfrak{P}_m \mid \mathcal{P}$ be a tower of places, $\mathcal{P} \subseteq K, \mathfrak{P}_m \subseteq M, \mathfrak{P} \subseteq L$, where M is an intermediate field of degree p over K . Then the number of jumps in the ramification group filtrations $G_i(\mathfrak{P}|\mathcal{P})$ is the number of distinct different exponents $d(\mathfrak{P}_m|\mathcal{P})$, where M runs through all intermediate fields of L/K of degree p over K .*

PROOF. Fix \mathfrak{P} and \mathcal{P} , and let M run through all possible degree p intermediate fields of L/K . By (6), t is equal to the number of distinct values of $d(\mathfrak{P}|\mathfrak{P}_m)$ since $d(\mathfrak{P}|\mathcal{P})$ and $e(\mathfrak{P}|\mathfrak{P}_m) = p^{n-1}$ are independent of the choice of M . For $0 \leq i \leq l-1$, the i -th jump occurs at index m_i ; that is, $G_{m_i} \not\supseteq G_{m_i+1}$.

By Lemma 4.2, there exists an order p^{n-1} subgroup H_i of G_0 , so that $H_i \supseteq G_{m_i+1}$ but $H_i \not\supseteq G_{m_i}$. By Lemma 4.1, we have $\#(H_i \cap G_{m_i}) = \frac{\#G_{m_i}}{p}$. Similarly, $\#(H_i \cap G_j) = \frac{\#G_j}{p}$ for all $j \leq m_i$ since G_j is decreasing. Hence, $\#(H_i \cap G_j) = \frac{\#G_j}{p}$ for $0 \leq j \leq m_i$, and $\#(H_i \cap G_j) = \#G_j$ for $j > m_i$. Let M_i be the intermediate field of L/K corresponding to H_i under Galois correspondence, and set $\mathfrak{P}_i = \mathfrak{P} \cap M_i$. Since $G_j(\mathfrak{P}|\mathfrak{P}_i) = H_i \cap G_j(\mathfrak{P}|\mathcal{P}) = H_i \cap G_j$ for all j , we have

$$(11) \quad d(\mathfrak{P}|\mathfrak{P}_i) = \sum_{j=0}^{\infty} (\#G_j(\mathfrak{P}|\mathfrak{P}_i) - 1) = \sum_{j=0}^{\infty} (\#(H_i \cap G_j) - 1) = \sum_{j=0}^{m_i} \left(\frac{\#G_j}{p} - 1 \right) + \sum_{j=m_i+1}^{\infty} (\#G_j - 1).$$

Note that the right-hand side of (11) is strictly decreasing with i . Hence, we find l pairwise distinct values of $d(\mathfrak{P}|\mathfrak{P}_m)$. Note that any order p^{n-1} subgroup H of G_0 contains $G_{m_i+1} = \{\text{Id}\}$ but not $G_{m_0} = G_0$. Hence, for any such H , there exists an i such that $H \supseteq G_{m_i+1}$ but $H \not\supseteq G_{m_i}$. In other words, H is one of the H_i by the previous analysis of the choice of H_i . Thus, there are exactly l pairwise distinct values of $d(\mathfrak{P}|\mathfrak{P}_m)$ when M runs on all possible degree p intermediate fields of L/K . Hence, there are exactly l pairwise distinct values of $d(\mathfrak{P}_m|\mathcal{P})$, i.e. $l = t$. \square

By Proposition 2.3, $d(\mathfrak{P}_m|\mathcal{P})$ is a multiple of $p-1$ for any M . Hence,

$$(12) \quad d_i = \frac{d(\mathfrak{P}_i|\mathcal{P})}{p-1} - 1$$

is an integer for all $0 \leq i \leq l-1$. By (11), $d(\mathfrak{P}|\mathfrak{P}_i)$ is strictly decreasing with i . Hence, d_i is strictly increasing with i by (12).

Now we are ready for the main result.

THEOREM 4.4. *Let L/K be a Galois extension of function fields whose Galois group is a p -group satisfying (5), where p is the characteristic of K . Let $\mathfrak{P}|\mathcal{P}$ be places, $\mathcal{P} \subseteq K$, $\mathfrak{P} \subseteq L$, m_i as in (10) for $0 \leq i$, and d_j as in (12) for $j \leq l-1$. Then $m_i = d_0 + \sum_{j=1}^i p^{n-n_j}(d_j - d_{j-1})$ for $0 \leq i \leq l-1$.*

PROOF. By applying (7) to L/K , we have

$$(13) \quad d(\mathfrak{P}|\mathcal{P}) = \sum_{j=0}^{l-1} (p^{n_j} - 1)(m_j - m_{j-1}),$$

where $m_{-1} = -1$. Since for $0 \leq i \leq l-1$ and $\#G_j = p^{n_i}$ for $m_{i-1} < j \leq m_i$, then (11) yields

$$(14) \quad d(\mathfrak{P}|\mathfrak{P}_i) = \sum_{j=0}^i (p^{n_j-1} - 1)(m_j - m_{j-1}) + \sum_{j=i+1}^{l-1} (p^{n_j} - 1)(m_j - m_{j-1}).$$

Now substituting (13), (14), $e(\mathfrak{P}|\mathfrak{P}_m) = p^{n-1}$, and (12) into (6) for the case $\mathfrak{P}_m = \mathfrak{P}_i$, we have

$$\sum_{j=0}^{l-1} (p^{n_j} - 1)(m_j - m_{j-1}) = p^{n-1}(d_i + 1)(p-1) + \sum_{j=0}^i (p^{n_j-1} - 1)(m_j - m_{j-1}) + \sum_{j=i+1}^{l-1} (p^{n_j} - 1)(m_j - m_{j-1}).$$

Hence, we have

$$\sum_{j=0}^i (p^{n_j} - 1)(m_j - m_{j-1}) = (p^n - p^{n-1})(d_i + 1) + \sum_{j=0}^i (p^{n_j-1} - 1)(m_j - m_{j-1}),$$

which implies

$$(15) \quad \sum_{j=0}^i (p^{n_j} - p^{n_j-1})(m_j - m_{j-1}) = (p^n - p^{n-1})(d_i + 1).$$

When $i = 0$, (15) yields

$$(p^n - p^{n-1})(m_0 + 1) = (p^{n_0} - p^{n_0-1})(m_0 + 1) = (p^n - p^{n-1})(d_0 + 1),$$

which implies $m_0 = d_0$. Thus, the formula in Theorem 4.4 is true when $i = 0$. Now we induct on i . By (15), we have

$$(p^{n_i} - p^{n_i-1})m_i - (p^{n_i} - p^{n_i-1})m_{i-1} + \sum_{j=0}^{i-1} (p^{n_j} - p^{n_j-1})(m_j - m_{j-1}) = (p^n - p^{n-1})(d_i + 1).$$

It follows that

$$(p^{n_i} - p^{n_{i-1}})m_i - (p^{n_i} - p^{n_{i-1}})m_{i-1} + (p^n - p^{n-1})(d_{i-1} + 1) = (p^n - p^{n-1})(d_i + 1)$$

by applying (15) to the case $i - 1$, which implies

$$(p^{n_i} - p^{n_{i-1}})m_i = (p^n - p^{n-1})(d_i - d_{i-1}) + (p^{n_i} - p^{n_{i-1}})m_{i-1}.$$

By induction hypothesis, it follows that

$$(p^{n_i} - p^{n_{i-1}})m_i = (p^n - p^{n-1})(d_i - d_{i-1}) + (p^{n_i} - p^{n_{i-1}})\left(d_0 + \sum_{j=1}^{i-1} p^{n-n_j}(d_j - d_{j-1})\right).$$

Dividing both sides by $p^{n_i} - p^{n_{i-1}}$, we then obtain

$$m_i = d_0 + p^{n-n_i}(d_i - d_{i-1}) + \sum_{j=1}^{i-1} p^{n-n_j}(d_j - d_{j-1}) = d_0 + \sum_{j=1}^i p^{n-n_j}(d_j - d_{j-1}).$$

Hence, our result follows by induction. \square

The formula in Theorem 4.4 can be reformulated to be easily compared to the Hasse-Arf property.

COROLLARY 4.5. *With notation as in Theorem 4.4, we have $m_i - m_{i-1} = p^{n-n_i}(d_i - d_{i-1})$, for $0 \leq i \leq l - 1$, where $m_{-1} = -1$.*

PROOF. Immediate by applying the formula in Theorem 4.4 to the cases i and $i - 1$. \square

CHAPTER V

HASSE-ARF PROPERTY

The formula in Corollary 4.5 is expected due to the well-known Hasse-Arf property, see [2]. It claims that the distance between two consecutive jumps in a ramification group filtration is divisible by the index of the group at the jump in the first group of the filtration. The Hasse-Arf property is true when the Galois group is abelian yet not always true otherwise.

In our setting, the Hasse-Arf property translates to $p^{n-n_j} \mid m_j - m_{j-1}$. So, according to Corollary 4.5, not only do we verify that it is true, we also know the quotient to be $d_i - d_{i-1}$. An advantage of knowing the quotient explicitly is that we can discuss whether the Hasse-Arf property can be improved or not. In fact, we can construct an example where the group index is a power of p , and no higher power of p can divide $m_j - m_{j-1}$ than the power guaranteed by the Hasse-Arf property. Notice that the strictly increasing property and $d_i \not\equiv 0 \pmod{p}$ are the only two restrictions on the sequence d_i of positive integers. See [1] or [22] for a discussion of the type of the extension L/K . Although an explicit construction is not given there, the extension L/K herein is of the same type as described in those two papers.

Actually, we can construct Artin-Schreier extensions M_i over the same base field K with any prescribed different exponent $(d_i + 1)(p - 1)$; then we can construct L to be the composite of those M_i . That is to say, for any strictly increasing sequence of nonnegative integers d_i of length l , we can construct a Galois extension L/K of function fields and corresponding extension of places \mathfrak{P} lying over \mathcal{P} , so that there are exactly l jumps in the ramification group filtration of $\mathfrak{P}|\mathcal{P}$ and exactly l pairwise distinct values of the different exponents $(d_0 + 1)(p - 1) < (d_1 + 1)(p - 1) < \dots < (d_{l-1} + 1)(p - 1)$ for $d(\mathfrak{P}_m|\mathcal{P})$ when M ranges over all degree p intermediate fields of L/K . In particular, we can require that $d_i = d_{i-1} + 1$ for all $1 \leq i \leq l - 1$. With this example, we know that there is no way to improve the Hasse-Arf divisibility result. On the other hand, $d_i - d_{i-1}$ can be any prescribed positive integer, so it is possible to strengthen the Hasse-Arf divisibility result arbitrarily under certain circumstances.

Now we want to analyze the Hasse-Arf property under a more general assumption; that is to say, to remove the totally wildly ramified assumption. First, we consider the not-totally-ramified case. With the notation as in Theorem 4.4 and M as the inertia field of the place extension $\mathfrak{P}|\mathcal{P}$, we know that $G_i(\mathfrak{P}|\mathcal{P}) = G_i(\mathfrak{P}|\mathfrak{P}_m)$ for all $i \geq 0$ by Lemma 3.1. Hence, the Hasse-Arf property is true for the partially ramified case with identical parameters and formulae to those in the totally ramified case.

However, the situation changes when we move to the tamely ramified case. For that purpose, let

$[L : K] = p^m q$ such that $p \nmid q$, M the intermediate field of L/K corresponding to a Sylow p -subgroup of $\text{Gal}(L/K)$ under Galois correspondence, and $\mathfrak{P}_m = \mathfrak{P} \cap M$. By Corollary 4.5, we have $m_i - m_{i-1} = p^{n-n_i}(d_i - d_{i-1})$ for $0 \leq i \leq l-1$, where m_i and d_i are defined for the ramification filtrations $G_i(\mathfrak{P}|\mathfrak{P}_m)$. By Proposition 3.2, we know $G_i(\mathfrak{P}|\mathcal{P}) = G_i(\mathfrak{P}|\mathfrak{P}_m)$ for all $i \geq 1$. Hence, the i -th jump in the ramification filtrations of $G_i(\mathfrak{P}|\mathcal{P})$ is equal to $p^{n-n_i}(d_i - d_{i-1})$ for $1 \leq i \leq l-1$.

Therefore, the tamely ramified case does not satisfy the Hasse-Arf property in general since the distance needs to be divisible by $p^{n-n_i}q$, not p^{n-n_i} . Noticeably, it violates the Hasse-Arf property simply because it has an unexpected leading element G_0 in the ramification filtration. Hence, this is a removable violation. An easy way to address this is to manually modify the group index assumption from G_0 to a Sylow p -subgroup of $\text{Gal}(L/K)$ under Galois correspondence. As a consequence, the Hasse-Arf property is true in the case that L/K is not necessarily assumed to be totally wildly ramified.

CHAPTER VI

CONCLUSION

Now we want to review our results briefly. We analyzed the ramification group filtrations of a Galois function field extension, and reduced the investigation to the totally wildly ramified case. It turns out that the result is explicit. An explanation of why we can obtain such an explicit formula as in Theorem 4.4 is as follows. From (15), we have exactly l linear equations for the l variables m_i for $0 \leq i \leq l - 1$. Since the coefficient matrix is triangular in addition to being non-singular, we can expect that the solution not only exists and is unique but also could be expressed explicitly.

Due to the explicit nature of Corollary 4.5, we can discuss the Hasse-Arf property of such extensions and explore whether it can be strengthened or not. The general answer is no, and there exist examples to show that Hasse-Arf is the best possible divisibility result. Although we can discuss the ramification groups under the totally wild ramified assumption without loss of generality, we discussed whether or not the Hasse-Arf property is true under the general assumptions. The answer is yes, but we have to slightly modify the formulation of the Hasse-Arf property to apply it to the tamely ramified case.

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